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CONTENTS

An Investigation of the Average Distribution of Twin Prime Numbers. *By Charles S. Sutton* 1

Remark on the Theory of Bending of Plates of Variable Thickness. *By M. Eric Reissner* 43

The Mapping of Graphs on Surfaces. *By I. N. Kagno* 46

Four Place Table of the Bessel Function. *By M. M. Clapp* 76

Reaction Velocity in the System: $Solid_I + Gas \rightarrow Solid_{II}$. *By George S. Gardner* 77

R-Matrices and Equivalent Networks. *By Richard S. Burington* 85

On Product Integration. *By Garrett Birkhoff* 104

A Table of Planck's Function from 3500° to 8000°K. *By Parry Moon* 133

On Certain Reductions in the Four Color Problem. *By C. E. Winn* 159

Note on the Four Color Problem. *By Philip Franklin* 172

A Theorem relating Non-vanishing and Analytic Functions. *By Norman Levinson* 185

A Theorem on Absolutely Convergent Trigonometrical Series. *By H. R. Pitt* 191

Solution of Laplace's harmonic equation with three independent variables. *By S. A. Savin* 196

On the Relation between Simple Distillation and True Boiling Point Curves of Complex Mixtures the Components of which obey Raoult's Law. *By E. M. J. Pease* 202

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AN INVESTIGATION OF THE AVERAGE DISTRIBUTION OF TWIN PRIME NUMBERS

By Charles S. Sutton*

It is the purpose of this thesis to study the average distribution of twin prime numbers, and in particular, to obtain a formula giving the approximate number of twin primes less than a given upper limit. This formula must be regarded as empirical, insofar as its derivation depends upon certain assumptions which, though they appear to be reasonable, cannot be rigorously justified. The technique employed depends upon probability considerations, and in this respect is similar to the method used by H. Cramér (Acta Arithmetica, vol. 2, 1936) to obtain certain suggested results concerning the difference between consecutive prime numbers.

The relation (24) of Chapter III had been derived before any statistical investigation of twin primes had been made, and the author was greatly pleased to find that this asymptotic formula agreed quite well with the actual count of twin primes less than N, being in error by only about one percent for large values of N. This seemed to indicate that the assumptions underlying this formula were not, at any rate, gravely in error.

The omission of one of the less fundamental assumptions and the necessary introduction of an empirical constant resulted in a formula which is considerably better than the first formula, as a glance at Tables IV and V will readily show.

TABLE OF CONTENTS

I. Introduction 2
II. Application of probability to primes 7
III. Application of probability to twin primes 13
IV. The more general case of pairs of primes differing by any integer 20
V. Comparison of values given by $f(N) = C_0 \cdot \int_2^N \frac{dt}{(\ln t)^2}$ with count of twin primes up to 800,000 31

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VI. The analogue of Riemann's formula for twin primes..... 35
VII. Conclusions..... 39

Appendix..... 40

I. Introduction

In the series of prime numbers, 1, 2, 3, 5, 7, 11, 13, 17, 19, 23, $\cdots$, there are certain pairs of primes which differ by two, such as 1, 3; 3, 5; 5, 7; 11, 13; 17, 19; etc. These pairs are commonly known as "twin primes". For the sake of simplicity, we shall designate each pair by the lower prime number in that pair, and accordingly define a twin prime number as any prime number which yields another prime when increased by two. Thus the sequence of twin primes will be: 1, 3, 5, 11, 17, 29, 41, $\cdots$. It is the purpose of this thesis to study the distribution of twin primes, and in particular to obtain a formula giving the approximate number of twin primes less than a given upper limit.

The question of the average distribution of prime numbers has long been the subject of investigation by mathematicians, and has finally been worked out to a remarkable degree of perfection. The distribution of twin primes, on the other hand, presents a much more difficult problem, and at present very little is definitely known on this subject. Before beginning our investigation of twin primes, however, we shall consider briefly what progress has been made in the study of prime numbers.

The earliest theorem of a general sort connected with primes is given by Euclid, who proves by a very elegant analysis that the number of primes is infinite. His argument is substantially as follows: Suppose that the number of primes were finite, and that there existed, therefore, a largest prime, P. Then the number $(P! + 1)$ cannot be divisible by any of the primes from 2 to P, since it gives a remainder of unity when divided by any such prime. It must therefore either be divisible by a prime greater than P or must itself be a prime (greater than P). In either event, the hypothesis that P is the largest prime is contradicted, and hence the number of primes cannot be finite.

Eratosthenes, a contemporary of Euclid, was the inventor of a "sieve" process for removing the composite numbers from the series of natural numbers. He first wrote the numbers in order, and then struck out every other number after 2, then every third number after 3, then every fifth number after 5, and so forth, rejecting in turn the multiples of the successive unerased numbers. After removing in this manner the

multiples of all primes less than any prime p, the series of prime numbers is complete as far as p^2.

The first approximative formula for the number of primes less than a given limit was given by Legendre in his "Théorie des Nombres", published in 1798. He noted by examining a table of primes that the number of primes up to 10 is equal to 5, the number up to 10^2 is $\frac{10^2}{4}$ very nearly, up to 10^3 is $\frac{10^3}{6}$ very nearly, up to 10^4 is $\frac{10^4}{8}$ very nearly, and up to 10^5 is $\frac{10^5}{10}$ very nearly. The simple formula $f(n) = \frac{n}{2 \log n}$, where $\log n$ is the logarithm to the base 10 of n, is fairly accurate, but is still in error by at least 4% if n exceeds 10,000. Legendre attempted to remedy this by considering the more general function $f(n) = \frac{n}{A \ln n + B}$, where $\ln n$ is the natural logarithm of n, and determining the constants A and B empirically by means of a count of prime numbers extending up to 400,000. He took A equal to 1 and found B to be equal to -1.08366. At the end of this chapter is given a table of values of Legendre's formula for various values of n less than 1,000,000,000, and the corresponding number of primes as determined from Lehmer's table of prime numbers (the last four entries are on the authority of Bertelsen and Meissel, Acta Mathematica, Vol. 17, p. 312). It will be observed that for values of n less than 1,000,000, the results agree fairly well, but that for larger values of n, the error gradually increases, becoming as great as 70,000 at the end of the table.

The next approximate formula for the number of primes less than a given upper limit was given in a theorem first conjectured by Dirichlet in 1838 and later by Gauss in 1849. It was not until 1898 that von Mangoldt was able to give a rigorous proof of this theorem, which may be stated as follows:

The number of primes less than n may be expressed by the logarithmic integral

$$Li(n) = \lim_{\epsilon \to 0} \left\{ \int_0^{1-\epsilon} \frac{dt}{\ln t} + \int_{1+\epsilon}^{n} \frac{dt}{\ln t} \right\} \tag{1}$$

with an error which may be made as small as desired in comparison with $Li(n)$. In other words, if $\pi(n)$ denote the number of primes less than n, then the ratio of $\pi(n)$ to $Li(n)$ tends to the limit unity as n increases without limit.

The convergence of this ratio to the limit unity is roughly illustrated by the following table:

TABLE I

n	$\pi(n)$	$Li(n)$	$\frac{\pi(n)}{Li(n)}$
1,000	168	178	0.94
10,000	1,229	1,246	0.98
100,000	9,592	9,630	0.996
1,000,000	78,498	78,628	0.9983
10,000,000	664,579	664,918	0.9994
100,000,000	5,761,455	5,762,209	0.99986
1,000,000,000	50,847,478	50,849,235	0.99996

When the logarithmic integral is successively integrated by parts it takes the form

$$Li(n) = \frac{n}{\ln n} + \frac{n}{(\ln n)^2} + \frac{2!\,n}{(\ln n)^3} + \cdots . \tag{2}$$

This is a divergent series, but the first term is the principal value of the integral, and so furnishes an asymptotic expression for the number of primes less than n. Hence we also have that the ratio of $\pi(n)$ to $\frac{n}{\ln n}$ tends to the limit unity as n increases without limit.

The question was still unanswered as to which of the two expressions, $\frac{n}{\ln n}$ or $Li(n)$, is the better or more exact formula. In 1898 de la Vallée Poussin was able to show that the logarithmic integral is an asymptotic expression for the number of primes which is more exact than any of its possible expressions in finite form, and that the error committed in using it cannot exceed $\frac{ane^{-b\sqrt{\ln n}}}{\ln n}$, where a and b are fixed positive numbers.

We can now see by expanding Legendre's function $f(n) = \frac{n}{A \ln n + B}$ in inverse powers of $\ln n$, and comparing with (2), that the least inaccurate formula of the Legendre type is obtained by taking $A = 1$ and $B = -1$. Even then the formula is certain to be in error by quantities of the order of $\frac{n}{(\ln n)^2}$.

The best formula which has yet been obtained for the number of

TABLE II

n	$\pi(n)$
50,000	5,134
100,000	9,593
150,000	13,849
200,000	17,985
250,000	22,045
300,000	25,998
350,000	29,978
400,000	33,861
450,000	37,707
500,000	41,539
600,000	49,099
700,000	56,544
800,000	63,952
900,000	71,275
1,000,000	78,499
1,500,000	114,156
2,000,000	148,934
2,500,000	183,073
3,000,000	216,817
3,500,000	250,151
4,000,000	283,147
4,500,000	315,949
5,000,000	348,514
5,500,000	380,801
6,000,000	412,850
6,500,000	444,758
7,000,000	476,649
7,500,000	508,262
8,000,000	539,778
8,500,000	571,120
9,000,000	602,490
9,500,000	633,579
10,000,000	664,580
20,000,000	1,270,608
90,000,000	5,216,955
100,000,000	5,761,456
1,000,000,000	50,847,479

primes less than a given limit is due to Riemann. This formula may be written

$$R(n) = A + Li(n) - \frac{1}{2}Li(n^{\frac{1}{2}}) - \frac{1}{3}Li(n^{\frac{1}{3}}) - \frac{1}{5}Li(n^{\frac{1}{5}}) + \cdots$$
$$= A + \sum_{k=1}^{\infty} \frac{\mu(k)Li(n^{1/k})}{k}, \tag{3}$$

where A is a certain constant, which may be omitted without appreciable effect. The function $\mu(k)$ is equal to 1 for $k = 1$, equal to zero for every

TABLE III

n	Deviations		
	Legendre	von Mangoldt	Riemann
50,000	+2	+33	−1
100,000	−5	+37	−6
150,000	−5	+44	−5
200,000	−3	+51	−3
250,000	−10	+49	−10
300,000	+26	+89	+25
350,000	−17	+49	−18
400,000	−7	+62	−9
450,000	+2	+74	0
500,000	−6	+67	−9
600,000	−3	+74	−8
700,000	+21	+101	+13
800,000	+3	+85	−7
900,000	+4	+87	−9
1,000,000	+44	+129	+29
1,500,000	+23	+107	−10
2,000,000	+42	+121	−10
2,500,000	+102	+172	+29
3,000,000	+96	+154	−1
3,500,000	+124	+168	+4
4,000,000	+176	+205	+32
4,500,000	+153	+165	−17
5,000,000	+130	+124	−65
5,500,000	+175	+151	−46
6,000,000	+271	+227	+23
6,500,000	+336	+272	+61
7,000,000	+263	+178	−39
7,500,000	+325	+218	−5
8,000,000	+350	+222	−7
8,500,000	+427	+276	+41
9,000,000	+360	+186	−54
9,500,000	+467	+269	+24
10,000,000	+560	+338	+87
20,000,000	+1,043	+297	−37
90,000,000	+5,989	+855	+227
100,000,000	+6,548	+753	+96
1,000,000,000	+70,029	+1,756	−24

value of k which contains a repeated prime factor, and equal to $+1$ or -1 according as k contains an even or odd number of distinct prime factors

(all different). A glance at the table at the end of this chapter (Table III) will make clear the wonderful faithfulness with which this formula represents $\pi(n)$. No one has as yet been able to prove that Riemann's function is necessarily a better expression for $\pi(n)$ than is $Li(n)$, but this certainly seems to be the case, at least for values of n less than a billion.

We give now a table of values of $\pi(n)$ (given in Table II), together with the corresponding deviations of the approximating functions of Legendre, von Mangoldt (the logarithmic integral), and Riemann (given in Table III).

We cannot hope to derive by rigorous methods any formula giving the approximate number of twin primes less than a given upper limit. Some of the world's leading number theorists have worked on this problem without success. The best we can do is to make certain simplifying assumptions, and see what can be made to follow from these assumptions. We shall see what can be said in a rough way about the distribution of primes, and then apply the same technique to the study of twin primes.

II. Application of Probability to Primes

Suppose we wished to determine the approximate number of primes between N and $(N + \Delta N + 1)$. There are altogether just ΔN integers in this interval. Hence, if we subtract from ΔN the number of composite numbers in this interval, we shall obtain the number of primes.

Now there are approximately $\frac{\Delta N}{2}$ multiples of 2 in this interval, and the error involved in this approximation cannot exceed unity. Hence there will be approximately $\left(\Delta N - \frac{\Delta N}{2}\right)$ numbers in this interval which are not divisible by 2. Similarly, we ought to reject the (approximately) $\frac{\Delta N}{3}$ multiples of 3. But in so doing, we are rejecting for the second time the multiples of 6, so that the number of these multiples—approximately $\frac{\Delta N}{6}$—ought to be added again. Hence the number of numbers in this interval which are divisible by neither 2 nor 3 will be approximately $\Delta N - \frac{\Delta N}{2} - \frac{\Delta N}{3} + \frac{\Delta N}{2 \cdot 3} = \Delta N\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)$. Since the error of each individual term in the sum cannot exceed unity, the

error of the entire expression must be less than 4, and hence the fractional error must be less than

$$\frac{4}{\Delta N\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)-4}$$

Similarly, we reject the (approximately) $\frac{\Delta N}{5}$ multiples of 5. In so doing, we are rejecting for the second time the multiples of 10 and 15, so that these must again be added. The multiples of 30 have been rejected three times (multiples of 2, 3, and 5) and added three times (multiples of 6, 10, and 15), so that these must again be rejected. Hence the number of numbers not divisible by either 2, 3, or 5 will be approximately

$$\Delta N-\frac{\Delta N}{2}-\frac{\Delta N}{3}-\frac{\Delta N}{5}+\frac{\Delta N}{2\cdot 3}+\frac{\Delta N}{2\cdot 5}+\frac{\Delta N}{3\cdot 5}-\frac{\Delta N}{2\cdot 3\cdot 5}$$
$$=\Delta N\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)$$

It is easily seen that the above form will continue to hold for any number of primes. The number of numbers not divisible by any of the primes 2, 3, 5, 7, $\cdots P$ will be approximately

$$\Delta N-\sum\frac{\Delta N}{p}+\sum_{p<q}\frac{\Delta N}{pq}-\sum_{p<q<r}\frac{\Delta N}{pqr}+\cdots$$
$$=\Delta N\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\cdots\left(1-\frac{1}{P}\right) \tag{4}$$

where p, q, r, etc. are primes less than or equal to P (excluding unity), and the prime factors in the denominators are all distinct. If P be the $(R+1)$th prime number, there will be 2^R terms in the above expanded product, and since the error involved in each term is less than unity, the fractional error of the above expression cannot exceed

$$\frac{2^R}{\Delta N\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\cdots\left(1-\frac{1}{P}\right)-2^R} \tag{5}$$

We see that, if P be held fixed, the fractional error may be made arbitrarily small by taking ΔN sufficiently large.

The expression (4) may be interpreted in a rather simple and inter-

esting way. Since every other number is divisible by 2, we may say that the probability that any number, chosen at random, is divisible by 2 is $\frac{1}{2}$; hence the probability that any number is not divisible by 2 is $\left(1 - \frac{1}{2}\right)$. Similarly, since every third number is divisible by 3, every fifth number is divisible by 5, and so on, we may say that the probability that any number is not divisible by 3 is $\left(1 - \frac{1}{3}\right)$, the probability that any number is not divisible by 5 is $\left(1 - \frac{1}{5}\right)$, and so forth. If we consider the probabilities of these events as being independent (i.e., if the fact that a certain number, chosen at random in the interval, is not divisible by a certain number of primes $p_1, p_2, \cdots, p_k$, does not affect the probability that it is not divisible by any other prime p'), then it is legitimate to multiply these separate probabilities together, obtaining

$$\theta(P) = \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right) \cdots \left(1 - \frac{1}{P}\right) \tag{6}$$

for the probability that any number in the interval is not divisible by any of the primes 2, 3, 5, 7, $\cdots P$. $\theta(P)$ may also be considered as the average density of numbers not divisible by any of the primes 2, 3, 5, 7, $\cdots P$. Multiplying this probability by the length of the interval, ΔN, we obtain

$$\Delta N\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right) \cdots \left(1 - \frac{1}{P}\right) \tag{7}$$

for the approximate number of numbers in this interval which are not divisible by any of the primes up to and including P. This agrees with the expression (4), which we have seen is certainly valid if P is sufficiently small compared to ΔN. Hence we may infer that it *is* legitimate to consider the separate probabilities that a number is not divisible by 2, 3, 5, $\cdots P$ as being independent, *provided* P is sufficiently small compared to ΔN.

Now any number is a prime if it is not divisible by any prime less than the square root of itself, excluding unity. If the expression given in (7) were correct for sufficiently large values of P, we could take P approximately equal to $\sqrt{N}$ and obtain the approximate number of primes in the interval between N and $(N + \Delta N + 1)$. However, we might equally well say that a number is prime if it is not divisible by any prime

less than itself, in which case we would take P approximately equal to N, obtaining a different value for the approximate number of primes in the interval. It is therefore evident that the probabilities that a number is not divisible by each of the primes 2, 3, 5, $\cdots$ P cannot be considered as independent if P is as large as the square root of N.

We will now assume that the average density of primes in the vicinity of N is given by the expression

$$\theta(P) = \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right) \cdots \left(1 - \frac{1}{P}\right),$$

where P is the greatest prime less than N^{Γ}, and Γ is an exponent to be determined, which we shall call the "exponent for prime numbers". Of course we know in advance that there *is* an average density function for primes, for such a function would be obtained by differentiating the logarithmic integral, which is asymptotic to the number of primes less than a given limit. If there are $\Delta\pi(n)$ primes in an interval of length Δn, then the actual density of primes in this interval will be $\frac{\Delta\pi(n)}{\Delta n}$. Now we know that $\pi(n)$, the number of primes less than n, is asymptotic to the logarithmic integral, which is in turn asymptotic to $\frac{n}{\ln n}$. Hence we may write that $\pi(n) = \frac{n}{\ln n}(1 + \epsilon_1)$, and also that $\pi(n + \Delta n) = \frac{n + \Delta n}{\ln (n + \Delta n)}(1 + \epsilon_2)$, where ϵ_1 and ϵ_2 approach zero as n tends to infinity. Then $\frac{\Delta\pi(n)}{\Delta n} = \frac{\pi(n + \Delta n) - \pi(n)}{\Delta n}$.

Now place $\Delta n = kn$, where k is positive. Then

$$\begin{aligned}
\frac{\Delta\pi(n)}{\Delta n} &= \frac{1}{kn}\left[\frac{(n + kn)(1 + \epsilon_2)}{\ln(n + kn)} - \frac{n(1 + \epsilon_1)}{\ln n}\right] \\
&= \frac{(1 + k)(1 + \epsilon_2)\ln n - (1 + \epsilon_1)[\ln n + \ln(1 + k)]}{k \ln n[\ln n + \ln(1 + k)]} \\
&= \frac{1 - \frac{\ln(1 + k)}{k \ln n} + \epsilon_2\left(1 + \frac{1}{k}\right) - \frac{\epsilon_1}{k}\left[1 + \frac{\ln(1 + k)}{\ln n}\right]}{\ln n\left[1 + \frac{\ln(1 + k)}{\ln n}\right]}
\end{aligned}$$

From this we see that if k be given any fixed value, other than zero, then the ratio $\frac{\Delta\pi(n)}{\Delta n}$ will be asymptotic to $\frac{1}{\ln n}$, the derivative of the

logarithmic integral. In other words, if $\Delta n = A \cdot n$, where A is a positive constant, then the actual density of primes in an interval of length Δn will be asymptotic to $\frac{1}{\ln n}$, and hence this quantity may be properly called the average density of primes in the vicinity of n.

If we place $\Delta N = A \cdot N$ in (5), we obtain an upper limit for the fractional error of the function $\theta(P)$ giving the average density of numbers not divisible by any of the primes up to and including P. This upper limit then becomes $\frac{2^R}{AN \cdot \theta(P) - 2^R}$, which will clearly tend to zero if $\frac{2^R}{AN \cdot \theta(P)}$ tends to zero. Since this will obviously tend to zero if P remains finite as N tends to infinity, we assume P tends to infinity with N.

If $\frac{2^R}{AN \cdot \theta(P)}$ is to tend to zero, then its logarithm must become negatively infinite; i.e., we must have

$$R \ln 2 - \ln N - \ln[\theta(P)] - \ln A \to -\infty \tag{8}$$

Now it has been proven by Landau[1] that, as $P \to \infty$,

$$\theta(P) = \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right) \cdots \left(1 - \frac{1}{P}\right) \sim \frac{e^{-c}}{\ln P} \tag{9}$$

where c is Euler's constant, equal to $0.57721566 \cdots$. It follows from (9) that $\ln[\theta(P)] = -c - \ln \ln P + \epsilon_1$, where $\epsilon_1 \to 0$ as P tends to infinity. Also, since P is the $(R+1)$th prime number, $R = \frac{P}{\ln P}(1 + \epsilon_2)$, where $\epsilon_2 \to 0$ as $P \to \infty$. Substituting these two relations in (8), we obtain

$$\frac{P \cdot \ln 2}{\ln P}(1 + \epsilon_2) - \ln N + \ln \ln P + c - \ln A - \epsilon_1 \to -\infty \tag{10}$$

Since $\ln \ln P = o\left(\frac{P}{\ln P}\right)$ as $P \to \infty$, (10) will be satisfied if $\frac{P}{\ln P} < \ln N$. Now if $k_N = (\ln N) \cdot (\ln \ln N)$, and $N > e^e$, then $\ln k_N = \ln \ln N + \ln \ln \ln N > \ln \ln N$, so that $\frac{k_N}{\ln k_N} < \ln N$. Hence if $P < k_N$, then $\frac{P}{\ln P} < \frac{k_N}{\ln k_N} < \ln N$, so that (10) will be satisfied if $P < (\ln N)(\ln \ln N)$.

[1] Edmund Landau, "Handbuch der Lehre von der Verteilung der Primzahlen," §36, p. 140.

Hence we may conclude that the function $\theta(P)$ is asymptotic to the average density, in the vicinity of N, of numbers not divisible by any of the primes up to and including P, provided that P is not greater than $(\ln N)(\ln \ln N)$.[2] Thus we see that we may correctly apply the theory of probability in eliminating multiples of 2, 3, 5, $\cdots$ P, provided P is sufficiently small compared to N. It appears that the theory breaks down when P becomes commensurate with any positive power of N. The reason for this is probably the fact that, in considering the separate probabilities that a number is not divisible by any prime p, we have not taken account of the fact that the product of all the prime factors of a number must be equal to the number itself. Thus, if a number is not divisible by any prime less than N^{α}, it necessarily follows that the number cannot be divisible by any prime between $N^{1-\alpha}$ and N. This latter fact has not been taken into consideration in applying the theory of probability to the determination of the density of numbers not divisible by any of the primes less than P. Hence it is apparent that the correct density must be less than the previously derived density $\theta(P)$, when $P \sim N^{\alpha}(\alpha > 0)$.

If the previously derived density function were correct, the average density of primes would be given by $\theta(P)$, with P approximately equal to $N^{\frac{1}{2}}$. Since this density is actually too large, it may be made smaller by continuing the product $\theta(P)$ still further, until P is equal to some power of N greater than $\frac{1}{2}$. Hence we should expect Γ, the exponent for primes, to be greater than $\frac{1}{2}$. We shall now determine this exponent.

We assume that the average density of primes in the vicinity of N is given by $\theta(P)$, where P is as nearly as possible equal to N^{Γ}. If we wish $\theta(P)$ to be asymptotic to the correct density, $\frac{1}{\ln N}$, we see from (9) that we must have $\frac{e^{-e}}{\ln P} \sim \frac{1}{\ln N}$, or $\ln P \sim e^{-e} \ln N$. But we have $\ln P \sim \Gamma \cdot \ln N$, and hence $\Gamma = e^{-e} = 0.56146 \cdots$.

The function $\theta(P) = \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right) \cdots \left(1 - \frac{1}{P}\right)$ then

[2] It is the author's conjecture that this condition may be replaced by

$$\lim_{N \to \infty} \left(\frac{\ln P}{\ln N}\right) = 0.$$

will give the density of numbers not divisible by any of the primes up to and including P, provided P is not greater than $(\ln N)(\ln \ln N)$. If P be taken as great as $\sqrt{N}$, to determine the density of primes, the function will no longer give the correct density.[2] However, the correct average density of primes may be obtained by taking P approximately equal to $N^{\frac{1}{3}}$.

III. Application of Probability to Twin Primes

Having applied the theory of probability to the study of prime numbers, we shall now see what can be said about the distribution of twin primes. We have seen how the sequence of prime numbers may be determined by a sieve process, by eliminating from the sequence of whole numbers the multiples of 2, 3, 5, $\cdots P$. The remaining numbers will then be all the prime numbers between P' and P'^2, where P' is the prime next greater than P. The "sieve" is shown graphically in Fig. 1. Starting with zero, a circle is placed over every other number, a square over every third number, a triangle over every fifth number, and a cross over every seventh number. The numbers which have no

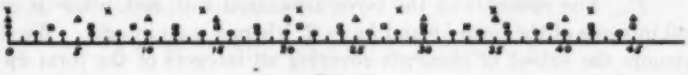

FIG. 1

symbol over them will then comprise the sequence of all numbers which are prime to 2, 3, 5, and 7 (i.e., which are not divisible by either 2, 3, 5, or 7), and in particular this sequence will coincide with the sequence of prime numbers between the limits 11 and 121.

Now we have defined a twin prime to be any prime which will yield another prime when increased by two. Hence, in order to obtain the sequence of twin primes, we must eliminate from the sequence of whole numbers not only those numbers which are divisible by any of the primes 2, 3, 5, $\cdots P$, but also those numbers which, when increased by two, will be divisible by 2, 3, 5, $\cdots P$. Hence the sieve for the determination of twin primes may be obtained by superimposing upon the sieve for prime numbers another sieve obtained by displacing that sieve two units to the left. Such a sieve is shown in Fig. 2, for $P = 7$. There is a circle over every number which is either divisible by 2 or will be divisible by 2 when increased by 2; there is a square over every number which is either divisible by 3 or will be divisible by 3 when increased by 2, and so forth. The numbers which have no symbol over them are the numbers which are neither divisible by 2, 3, 5, or 7, nor will be

divisible by 2, 3, 5, or 7 when increased by 2. This sequence will coincide with the sequence of twin primes between the limits 11 and 121.

We shall now show that the number of numbers, in an interval of length ΔN, which are neither divisible by any of the primes $2, 3, 5, \cdots P$, nor become divisible by any of these primes when increased by 2, is given approximately by

$$\Delta N\left(1-\frac{1}{2}\right)\left(1-\frac{2}{3}\right)\left(1-\frac{2}{5}\right)\cdots\left(1-\frac{2}{P}\right) \tag{11}$$

provided P is sufficiently small compared to ΔN. The significance of this product from a probability standpoint is evident by looking at Fig. 2.

The sieve for twin primes may be thought of as a superposition of separate layers, one layer associated with each of the primes 2, 3, 5,

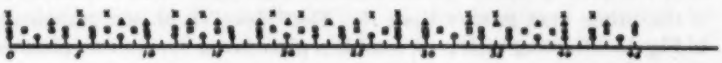

FIG. 2

$\cdots P$. The elements of the layer associated with any prime p cover all integers of the form kp and $kp-2$, where k is an integer. We shall denote the subset of elements covering all integers of the form kp by S_p^+, and the subset of elements covering all integers of the form $kp-2$ by S_p^-. We shall now prove that for any group of subsets $S_{p_1}^{\pm}, S_{p_2}^{\pm}, S_{p_3}^{\pm}, \cdots S_{p_n}^{\pm}$, where any arbitrary combination of signs may be taken and the p_i are all different, there will always be integral values of N such that N is covered by an element of each subset, and that these values of N will form an arithmetic progression with common difference equal to $p_1p_2p_3\cdots p_n$. No generality is lost by assuming the first m of these subsets to have a $(+)$ superscript, and the last $(n-m)$ subsets to have a $(-)$ superscript, so that we have to consider the subsets $S_{p_1}^+, S_{p_2}^+, S_{p_3}^+, \cdots, S_{p_m}^+, S_{p_{m+1}}^-, S_{p_{m+2}}^-, \cdots S_{p_n}^-$. We have then to solve the set of equations

$$N = k_1p_1 = k_2p_2 = \cdots = k_mp_m = k_{m+1}p_{m+1} - 2 = \cdots = k_np_n - 2 \tag{12}$$

for integral values of $N, k_1, k_2, k_3, \cdots, k_n$. Now the equations $N = k_1p_1 = k_2p_2 = \cdots = k_mp_m$ are clearly satisfied if and only if N is a multiple of each of the primes $p_1, p_2, \cdots p_m$. Hence we must have $N = a\cdot p_1p_2\cdots p_m$, where a is an integer. Also, the equations $N = k_{m+1}p_{m+1} - 2 = k_{m+2}p_{m+2} - 2 = \cdots = k_np_n - 2$ require that $(N+2)$

be a multiple of each of the primes $p_{m+1}, p_{m+2}, \cdots p_n$, so that we must have also $N = b \cdot p_{m+1}p_{m+2}p_{m+3} \cdots p_n - 2$. Hence the solution of (12) reduces to the solution of the equations

$$N = a \cdot p_1 p_2 \cdots p_m = b \cdot p_{m+1}p_{m+2} \cdots p_n - 2$$

for integral values of N, a, and b. Since N must necessarily be integral if a and b are integral, it suffices to consider the single equation

$$a(p_1 p_2 \cdots p_m) = b(p_{m+1}p_{m+2} \cdots p_n) - 2 \tag{13}$$

Now it is known from the theory of indeterminate equations that, since $(p_1 p_2 \cdots p_m)$ and $(p_{m+1}p_{m+2} \cdots p_n)$ are relatively prime (i.e., they contain no factor in common), this equation must have at least one solution, say $a = a_0$ and $b = b_0$, and that the general solution will then be

$$a = a_0 + j(p_{m+1}p_{m+2} \cdots p_n)$$

$$b = b_0 + j(p_1 p_2 \cdots p_m)$$

where j can have any integral value.[3] Since $N = a \cdot p_1 p_2 \cdots p_m$, we have for the general solution of (12)

$$N = [a_0 + j(p_{m+1}p_{m+2} \cdots p_n)] \cdot p_1 p_2 \cdots p_m$$

or

$$N = a_0 p_1 p_2 \cdots p_m + j(p_1 p_2 p_3 \cdots p_n) \tag{14}$$

This shows that the values of N lie in an arithmetic progression with common difference equal to $p_1 p_2 p_3 \cdots p_n$. Hence, in an interval of length ΔN, there will be approximately $\dfrac{\Delta N}{p_1 p_2 p_3 \cdots p_n}$ integers which are covered by elements from each of the subsets $S^+_{p_1}, S^+_{p_2}, S^+_{p_3}, \cdots S^+_{p_n}$, where the superscripts are all $(+)$. If ΔN is a multiple of $p_1 p_2 \cdots p_n$, this expression will be *exact*, and in all events the error involved cannot exceed unity. The same will hold true no matter what the superscripts may be, so long as the subscripts are $p_1, p_2, p_3, \cdots p_n$. Now, if $p \neq 2$, the subsets S^+_p and S^-_p are mutually exclusive, and if $p = 2$, the two subsets are coincident. Hence, the number of distinct combinations of n subsets, one taken from each of the layers associated with the primes $p_1, p_2, p_3, \cdots p_n$, will be 2^{n-1} or 2^n according as 2 is or is not among the primes $p_1, p_2, p_3, \cdots p_n$. Since each one of these combinations will recur every $(p_1 p_2 p_3 \cdots p_n)$ integers, the number of integers, in an interval of length ΔN, which are covered by elements

[3] See H. B. Fine, "College Algebra", p. 342.

from each of the layers associated with the primes $p_1, p_2, \cdots p_n$, will be approximately $\frac{2^n \cdot \delta_n \Delta N}{p_1 p_2 p_3 \cdots p_n}$, where the quantity δ_n is equal to $\frac{1}{2}$ if one of the factors in the denominator is 2, and is equal to unity otherwise. The error involved in this approximation will be zero if ΔN is a multiple of $p_1 p_2 \cdots p_n$, and in all events cannot exceed 2^n.

We can now determine the approximate number of integers, in an interval of length ΔN, which are covered by elements of the sieve. We assume the sieve to consist of layers associated with the primes 2, 3, 5, 7, $\cdots P$. It is easy to show that, if ΔN is a multiple of $2 \cdot 3 \cdot 5 \cdot 7 \cdots P$, then every integer which is covered by one or more elements of the sieve is counted exactly once in the series

$$\sum \frac{2\delta_1 \Delta N}{p} - \sum_{p<q} \frac{2^2 \delta_2 \Delta N}{pq} + \sum_{p<q<r} \frac{2^3 \delta_3 \Delta N}{pqr} - \cdots \qquad (15)$$

where $p, q, r, \cdots$, are primes not greater than P, and the prime factors in the denominators are all distinct.

Suppose an integer in the interval is covered by an element of only one layer of the sieve; then that integer is counted once in (15), namely as one of the (exactly) $\frac{2\delta_1 \Delta N}{p}$ numbers in this interval which are covered by elements of the layer associated with the prime p. Next suppose an integer in this interval is covered by elements of r layers associated with r of the primes 2, 3, 5, $\cdots P$. Then that integer will be counted exactly r times in $\sum \frac{2\delta_1 \Delta N}{p}$; it will be counted in $\sum_{p<q} \frac{2^2 \delta_2 \Delta N}{pq}$ a number of times equal to the number of combinations of r things taken 2 at a time, or $\binom{r}{2}$; it will be counted $\binom{r}{3}$ times in $\sum_{p<q<r} \frac{2^3 \delta_3 \Delta N}{pqr}$, and so on. Hence the total number of times that integer is counted will be

$$r - \binom{r}{2} + \binom{r}{3} - \cdots + (-1)^{r-1} \binom{r}{r} = 1 - (1-1)^r = 1$$

Hence, if ΔN is a multiple of $2 \cdot 3 \cdot 5 \cdot 7 \cdots P$, then the number of integers in that interval which are covered by elements of the sieve will be exactly given by (15). Each individual term in (15) is of the form $\frac{2^n \delta_n \Delta N}{p_1 p_2 p_3 \cdots p_n}$, and gives the number of integers which are covered by elements from each of the layers associated with the primes $p_1, p_2,$

$p_2, \cdots p_n$. But even if ΔN is not a multiple of $p_1 p_2 \cdots p_n$, the error involved in each term must be less than 2^n. Now, if the sieve consists of R layers (that is, if there are R primes, excluding unity, not greater than P), then there will be R terms in the first summation in (15), there will be $\binom{R}{2}$ terms in the second summation, $\binom{R}{3}$ in the third, and so on. Hence, no matter what ΔN may be, the absolute error of the expression (15) cannot exceed

$$R \cdot 2 + \binom{R}{2} \cdot 2^2 + \binom{R}{3} \cdot 2^3 + \cdots + \binom{R}{R} \cdot 2^R = (1+2)^R - 1 < 3^R \quad (16)$$

Since there are altogether just ΔN integers in the interval, the number of integers in the interval which are not covered by any elements of the sieve will be approximately

$$\Delta N - \Delta N \sum \frac{2\delta_1}{p} + \Delta N \sum_{p<q} \frac{2^2 \delta_2}{pq} - \Delta N \sum_{p<q<r} \frac{2^3 \delta_3}{pqr} + \cdots$$
$$= \Delta N \left(1 - \frac{1}{2}\right)\left(1 - \frac{2}{3}\right)\left(1 - \frac{2}{5}\right)\left(1 - \frac{2}{7}\right) \cdots \left(1 - \frac{2}{P}\right) \quad (17)$$

with an error less than 3^R. Hence the fractional error of (17) must be less than

$$\frac{3^R}{\Delta N \left(1 - \frac{1}{2}\right)\left(1 - \frac{2}{3}\right)\left(1 - \frac{2}{5}\right) \cdots \left(1 - \frac{2}{P}\right) - 3^R} \quad (18)$$

and we see that, as in the case of primes, the fractional error may be made arbitrarily small by taking ΔN sufficiently large in comparison to P.

Hence we may say that the average density of integers which are neither divisible by any of the primes 2, 3, 5, 7, $\cdots P$, nor become divisible by any of these primes when increased by 2, will be approximately

$$\phi(P) = \left(1 - \frac{1}{2}\right)\left(1 - \frac{2}{3}\right)\left(1 - \frac{2}{5}\right) \cdots \left(1 - \frac{2}{P}\right) \quad (19)$$

Noting the similarity in form between the different corresponding expressions we have derived relating to primes and to twin primes, it seems natural to suppose that the average density of twin primes will be given by $\phi(P) = \left(1 - \frac{1}{2}\right)\left(1 - \frac{2}{3}\right)\left(1 - \frac{2}{5}\right) \cdots \left(1 - \frac{2}{P}\right)$, where P

is the greatest prime less than N^{γ_2}, and γ_2 is a certain constant exponent, which we shall call the "exponent for twin primes".

The simplest and most obvious assumption that we can make regarding this exponent is that it is equal to the exponent for prime numbers; i.e., we assume $\gamma_2 = \Gamma$. Then, if we compare the average density of primes with the average density of twin primes in the vicinity of N, we will have $N^{\gamma_2} = N^{\Gamma}$, so that the values of P in $\theta(P)$ and $\phi(P)$ will be equal. If $\rho(N)$ and $\rho_2(N)$ denote the average densities of primes and twin primes in the vicinity of N, respectively, then

$$[\rho(N)]^2 \sim [\theta(P)]^2 = \left(1 - \frac{1}{2}\right)^2\left(1 - \frac{1}{3}\right)^2\left(1 - \frac{1}{5}\right)^2 \cdots \left(1 - \frac{1}{P}\right)^2$$

$$\rho_2(N) \sim \phi(P) = \left(1 - \frac{1}{2}\right)\left(1 - \frac{2}{3}\right)\left(1 - \frac{2}{5}\right) \cdots \left(1 - \frac{2}{P}\right)$$

$$\frac{\rho_2(N)}{[\rho(N)]^2} \sim 2\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{4^2}\right)\left(1 - \frac{1}{6^2}\right) \cdots \left[1 - \frac{1}{(P-1)^2}\right] \qquad (20)$$

The product (20) evidently converges, for P becoming infinite, to a certain constant value, which we shall denote by C_0. Then, as $N \to \infty$, $\rho_2(N) \sim C_0 \cdot [\rho(N)]^2$.

Now let $\pi(N)$ be a function giving the approximate number of primes less than N, and let $\pi_2(N)$ be a function giving the approximate number of twin primes less than N. Then $\frac{d\pi(N)}{dN}$ and $\frac{d\pi_2(N)}{dN}$ will represent, in a sense, the average density of primes and twin primes, respectively, in the vicinity of N, so that we shall place

$$\rho(N) = \frac{d\pi(N)}{dN} = \pi'(N) \text{ and } \rho_2(N) = \frac{d\pi_2(N)}{dN} = \pi_2'(N).$$

We then have

$$\pi_2'(N) \sim C_0 \cdot [\pi'(N)]^2 \qquad (21)$$

Since we know that the average density of primes is asymptotically given by the derivative of the logarithmic integral, we have $\pi'(N) \sim \frac{1}{\ln N}$, and hence (21) becomes

$$\pi_2'(N) \sim \frac{C_0}{(\ln N)^2} \qquad (22)$$

Now,

$$\pi_2(N) = \pi_2(2) + \int_2^M \pi_2'(t)\,dt + \int_M^N \pi_2'(t)\,dt \qquad (23)$$

By (22) we can, given any positive ϵ, choose M sufficiently large so that, for all values of N greater than M,

$$\frac{C_0(1-\epsilon)}{(\ln N)^2} < \pi_2'(N) < \frac{C_0(1+\epsilon)}{(\ln N)^2}$$

Hence $\int_M^N \pi_2'(t)\,dt = C_0(1+\delta)\cdot\int_M^N \frac{dt}{(\ln t)^2}$, where $|\delta| < \epsilon$. Equation (23) may then be written

$$\pi_2(N) = \pi_2(2) + \int_2^M \pi_2'(t)\,dt + C_0(1+\delta)\int_M^N \frac{dt}{(\ln t)^2}$$

or,

$$\begin{aligned}\pi_2(N) &= \pi_2(2) + C_0\int_2^N \frac{dt}{(\ln t)^2} + C_0\delta\int_2^N \frac{dt}{(\ln t)^2} \\ &\quad - C_0(1+\delta)\int_2^M \frac{dt}{(\ln t)^2} + \int_2^M \pi_2'(t)\,dt \\ &= \pi_2(2) + C_0 I_1 + C_0\delta I_1 - C_0(1+\delta)\cdot I_2 + I_3 \\ &= C_0 I_1\left[1 + \delta + \frac{\pi_2(2) - C_0(1+\delta)I_2 + I_3}{C_0 I_1}\right]\end{aligned}$$

Now, having fixed ϵ, and hence M, the values of I_2 and I_3 are also fixed; also, since I_1 tends to infinity with N, we may choose N sufficiently large so that

$$\left|\frac{\pi_2(2) - C_0(1+\delta)\cdot I_2 + I_3}{C_0 I_1}\right| < \epsilon$$

Since $|\delta| < \epsilon$, we have finally, for N sufficiently large,

$$C_0 I_1(1-2\epsilon) < \pi_2(N) < C_0 I_1(1+2\epsilon)$$

and hence

$$\pi_2(N) \sim C_0\cdot\int_2^N \frac{dt}{(\ln t)^2} \tag{24}$$

The above asymptotic relation has been derived on the assumption that $\gamma_2 = \Gamma$. If, on the other hand, $\gamma_2 \neq \Gamma$, then let P and P' be the greatest primes less than N^{γ_2} and N^{Γ}, respectively. Then $[\rho(N)]^2 \sim [\theta(P')]^2$ and $\rho_2(N) \sim \phi(P)$, so that

$$\frac{\rho_2(N)}{[\rho(N)]^2} \sim \frac{\phi(P)}{[\theta(P)]^2}\cdot\frac{[\theta(P)]^2}{[\theta(P')]^2} \sim C_0\cdot\left[\frac{\theta(P)}{\theta(P')}\right]^2$$

Now, by (9),

$$\left[\frac{\theta(P)}{\theta(P')}\right]^2 \sim \left[\frac{\ln P'}{\ln P}\right]^2 \sim \frac{\Gamma^2}{\gamma_2^2}$$

and hence

$$\rho_2(N) \sim C_0 \cdot \frac{\Gamma^2}{\gamma_2^2} \cdot [\rho(N)]^2 \tag{25}$$

Thus, if γ_2 is not equal to Γ, but has some other constant value (other than zero), the only effect will be to multiply the density of twin primes by a constant factor. We will then have, in place of (24), the relation

$$\pi_2(N) \sim C_0 \cdot \frac{\Gamma^2}{\gamma_2^2} \cdot \int_2^N \frac{dt}{(\ln t)^2} \tag{26}$$

We may therefore conclude that it seems extremely likely that the correct expression giving the approximate number of twin primes less than a given upper limit N is a certain constant times the integral $\int_2^N \frac{dt}{(\ln t)^2}$. (The lower limit is quite arbitrarily taken equal to 2; it might equally well be any other constant greater than unity).

IV. The More General Case of Pairs of Primes Differing by any Integer

The arguments of the preceding chapter may easily be generalized to include the more general case of pairs of primes differing by any integer k. If, for convenience, we designate each such pair by the lower prime number in that pair, we have to determine the density of those primes which will yield other primes when increased by k. The sieve for the determination of these primes will be obtained by superimposing upon the sieve for prime numbers another sieve obtained by displacing that sieve k units to the left. As before, we shall denote the subset of elements covering all integers of the form mp (m is an integer) by S_p^+, and the subset of elements covering all integers of the form $(mp - k)$ by S_p^-.

The argument of the preceding chapter will suffice to show that, in an interval of length ΔN, there will be approximately $\frac{\Delta N}{p_1 p_2 p_3 \cdots p_n}$ integers which are covered by elements from each of the subsets $S_{p_1}^+$, $S_{p_2}^+$, $S_{p_3}^+$, $\cdots$ $S_{p_n}^+$, where the superscripts are all (+), and that the

same will hold true no matter what the superscripts may be, so long as the subscripts are $p_1, p_2, p_3, \cdots p_n$. Now, if p is not a factor of k, the subsets S_p^+ and S_p^- are mutually exclusive, and if p is a factor of k, the two subsets are coincident. Hence, the number of distinct combinations of n subsets, one taken from each of the layers associated with the primes $p_1, p_2, p_3, \cdots p_n$, will be 2^n if none of the primes $p_1, p_2, p_3, \cdots p_n$ divide k, and will in general be 2^{n-s} if k is divisible by s of the primes $p_1, p_2, p_3, \cdots p_n$. Since each one of these combinations will recur every $(p_1 p_2 p_3 \cdots p_n)$ integers, the number of integers, in an interval of length ΔN, which are covered by elements from each of the layers associated with the primes $p_1, p_2, p_3, \cdots p_n$, will be approximately

$$\frac{2^n \cdot \delta_{p_1 k} \cdot \delta_{p_2 k} \cdot \delta_{p_3 k} \cdots \delta_{p_n k}}{p_1 p_2 p_3 \cdots p_n}$$

where δ_{pk} is equal to $\frac{1}{2}$ if p is a factor of k, and equal to unity otherwise.

As before, the number of integers in the interval of length ΔN which are not covered by elements of the sieve will be approximately

$$\Delta N - \Delta N \sum \frac{2\delta_{pk}}{p} + \Delta N \sum_{p<q} \frac{2^2 \delta_{pk}\delta_{qk}}{pq} - \Delta N \sum_{p<q<r} \frac{2^3 \delta_{pk}\delta_{qk}\delta_{rk}}{pqr} + \cdots$$
$$= \Delta N \left(1 - \frac{2\delta_{2k}}{2}\right)\left(1 - \frac{2\delta_{3k}}{3}\right)\left(1 - \frac{2\delta_{5k}}{5}\right) \cdots \left(1 - \frac{2\delta_{Pk}}{P}\right)$$

with an absolute error less than 3^R, where R is the number of layers in the sieve.

Hence we may say that the average density of integers which are neither divisible by any of the primes $2, 3, 5, \cdots P$, nor become divisible by any of these primes when increased by k, will be approximately

$$\phi_k(P) = \left(1 - \frac{2\delta_{2k}}{2}\right)\left(1 - \frac{2\delta_{3k}}{3}\right)\left(1 - \frac{2\delta_{5k}}{5}\right) \cdots \left(1 - \frac{2\delta_{Pk}}{P}\right) \quad (27)$$

We assume as before that the average density of primes which will yield other primes when increased by k will be given by $\phi_k(P)$, where P is taken to be the greatest prime less than N^{γ_k}, and γ_k is the exponent for pairs of primes differing by k. The function $\phi(P)$, as previously defined for twin primes, will then be equal to $\phi_2(P)$.

Now let $\rho_k(N)$ denote the average density of pairs of primes differing by k. Then

$$\rho_k(N) = \prod_{p=2}^{p \le N^{\gamma_k}} \left(1 - \frac{2\delta_{pk}}{p}\right)$$

But obviously δ_{pk} must equal unity if $p > k$, so that we may write, for $k \geq 3$,

$$\rho_k(N) = \prod_{p=2}^{p\leq k}\left(1 - \frac{2\delta_{pk}}{p}\right)\cdot \prod_{p>k}^{p\leq N^{\gamma_2}}\left(1 - \frac{2}{p}\right)$$

$$\rho_2(N) = \frac{1}{2}\cdot\prod_{p=3}^{p\leq k}\left(1 - \frac{2}{p}\right)\cdot \prod_{p>k}^{p\leq N^{\gamma_2}}\left(1 - \frac{2}{p}\right)$$

and hence

$$\frac{\rho_k(N)}{\rho_2(N)} = 2(1 - \delta_{2k})\cdot\prod_{p=3}^{p\leq k}\left(\frac{p - 2\delta_{pk}}{p - 2}\right)\cdot \frac{\prod_{p>k}^{p\leq N^{\gamma_k}}\left(1 - \frac{2}{p}\right)}{\prod_{p>k}^{p\leq N^{\gamma_2}}\left(1 - \frac{2}{p}\right)} \tag{28}$$

Now, from (9),

$$\frac{1}{2}\prod_{p=3}^{p=P}\left(1 - \frac{2}{p}\right) = \phi(P) \sim C_0\cdot[\theta(P)]^2 \sim C_0\cdot\frac{e^{-2c}}{(\ln P)^2}$$

and hence

$$\frac{\prod_{p>k}^{p\leq N^{\gamma_k}}\left(1 - \frac{2}{p}\right)}{\prod_{p>k}^{p\leq N^{\gamma_2}}\left(1 - \frac{2}{p}\right)} \sim \frac{\gamma_2^2}{\gamma_k^2}$$

Substituting this in (28) gives

$$\frac{\rho_k(N)}{\rho_2(N)} \sim 2\cdot\frac{\gamma_2^2}{\gamma_k^2}(1 - \delta_{2k})\cdot\prod_{p=3}^{p\leq k}\left(\frac{p - 2\delta_{pk}}{p - 2}\right) \tag{29}$$

Now, $2(1 - \delta_{pk})$ is equal to unity if p divides k, and is equal to zero otherwise. Setting $\Delta_{pk} = 2(1 - \delta_{pk})$, we have

$$\frac{p - 2\delta_{pk}}{p - 2} = 1 + \frac{2(1 - \delta_{pk})}{p - 2} = 1 + \frac{\Delta_{pk}}{p - 2},$$

so that (29) may be written

$$\frac{\rho_k(N)}{\rho_2(N)} = \frac{\gamma_2^2}{\gamma_k^2}\cdot\Delta_{2k}\cdot\prod_{p=3}^{p\leq k}\left(1 + \frac{\Delta_{pk}}{p - 2}\right) \tag{30}$$

where $\Delta_{pk} = 1$ if p divides k, and $\Delta_{pk} = 0$ otherwise. This formula holds also for $k = 1, 2$, if we interpret $\prod_{p=3}^{p\leq k}$ as being equal to unity for $k < 3$.

Now we have shown that the average density of primes in the interval $(N, N + \Delta N)$ is asymptotic to $\rho(N) \sim \frac{1}{\ln N}$, provided ΔN is of the order of magnitude of $A \cdot N$, where A is a fixed quantity greater than zero. Hence we may say that the probability that any number, chosen at random in this interval, will be prime, is $\frac{1}{\ln N}$, and hence the probability that any two numbers, chosen at random in this interval, will both be prime, is $\frac{1}{(\ln N)^2}$. If pairs of primes differing by two were just as dense as pairs of primes differing by any other integer, we should expect the density of such pairs to be just $\frac{1}{(\ln N)^2}$. We have seen, however, that these pairs are not all equally dense; in fact equation (30) shows that the density of all pairs of primes differing by any odd integer must be zero. Nevertheless we should expect, at any rate, that the *average* value (if it exists) of all the densities $\rho_k(N)$ would be asymptotically equal to $\frac{1}{(\ln N)^2} \sim [\rho(N)]^2$. We proceed to justify this statement.

We have seen that the average density of pairs of primes differing by k is determined by the superposition of two sieves for the determination of prime numbers, the second sieve being simply the first sieve shifted k units to the left. We also know that the average density of primes in the vicinity of $N + \theta N$, where $0 \leq \theta \leq 1$, is asymptotic to the average density of primes in the vicinity of N, for $\rho(N + \theta N) \sim \frac{1}{\ln(N + \theta N)} \sim \frac{1}{\ln N} \sim \rho(N)$. Hence the average character of the sieve for prime numbers (i.e., the average density of its elements) in the vicinity of $N + \theta N$ will be asymptotic to the average character of the sieve in the vicinity of N. But the sieve in the vicinity of $N + \theta N$ is identical with a new sieve in the vicinity of N, this new sieve being obtained by shifting the first sieve θN units to the left. Hence it seems reasonable to assume that, if $k \leq \theta N \leq N$, and $\Delta N = A \cdot N$, then the average density, in the interval $(N, N + \Delta N)$, of pairs of primes differing by k, will be asymptotic to $\rho_k(N)$, as N tends to infinity.

Now consider an interval of length ΔN, where $\Delta N = A \cdot N$, A being a positive constant greater than zero. Suppose we pick out two numbers at random in this interval, *specifying only* that they differ by less than $\epsilon \cdot \Delta N$, where ϵ is a fixed positive quantity. The number of pairs

of numbers, in the interval ΔN, which differ by k, will be $(\Delta N - k + 1) = \Delta N\left(1 - \frac{k-1}{\Delta N}\right)$. Since $k < \epsilon \cdot \Delta N$, $\frac{k-1}{\Delta N} < \epsilon$. Hence it is equally likely (to within a factor differing from unity by less than ϵ) that these two numbers differ by any of the integers 1, 2, 3, 4, $\cdots$ $[\epsilon \cdot \Delta N]$, where $[\epsilon \cdot \Delta N]$ denotes the greatest integer contained in $\epsilon \cdot \Delta N$. But, as assumed above, the probability that two numbers differing by k ($k \leq [\epsilon \cdot \Delta N]$ and $k \geq 1$) will both be prime is $\rho_k(N)$, and hence the probability that two numbers chosen at random and differing by less than $\epsilon \cdot \Delta N$ (but not coinciding) will both be prime is asymptotic to

$$\frac{\Delta N(\rho_1 + \rho_2 + \rho_3 + \cdots + \rho_{[\epsilon \Delta N]}) \cdot [1 + f(N, \epsilon)]}{[\epsilon \Delta N] \cdot \left\{\Delta N - \frac{1}{2}[\epsilon \Delta N] + \frac{1}{2}\right\}} \tag{31}$$

where $|f(N, \epsilon)|$ is less than ϵ. The numerator is the approximate number of pairs of primes differing by less than $\epsilon \Delta N$ (but not coinciding), and the denominator is the total number of distinct number pairs, differing by less than $\epsilon \Delta N$, in the interval.[4]

Now, if $a_1 N_1 \leq \delta N_1 \leq a_2 N_1$, where a_1 and a_2 are fixed positive constants, then the number of primes in the interval $(N_1, N_1 + \delta N_1)$ will be asymptotic to $\frac{\delta N_1}{\ln N_1}$, as N_1 tends to infinity. Hence the probability that any two distinct numbers, chosen at random in this interval, will both be prime, will be asymptotic to $\frac{1}{(\ln N_1)^2}$.[5] Now let the interval $(N_1, N_1 + \delta N_1)$ be any interval of length $\epsilon \cdot \Delta N$ included in the larger interval $(N, N + \Delta N)$. Then we will have $\delta N_1 = \epsilon \cdot \Delta N = \epsilon \cdot AN$, and $N \leq N_1 \leq N + \Delta N - \epsilon \cdot \Delta N$. Hence $N_1 - \Delta N \leq N \leq N_1$, so that $\delta N_1 = \epsilon \cdot AN \leq \epsilon \cdot AN_1$. Also, $\Delta N = AN \leq AN_1$, so that

$$\delta N_1 = \epsilon \cdot AN \geq \epsilon A(N_1 - \Delta N) \geq \epsilon A(N_1 - AN_1) = \epsilon AN_1(1 - A).$$

[4] The denominator is

$$\Delta N + (\Delta N - 1) + (\Delta N - 2) + \cdots + [\Delta N - [\epsilon \Delta N] + 1] = \frac{\Delta N(\Delta N + 1)}{2}$$

$$- \frac{[\Delta N - [\epsilon \Delta N]][\Delta N - [\epsilon \Delta N] + 1]}{2} = \frac{1}{2}[2\Delta N[\epsilon \Delta N] + [\epsilon \Delta N] - [\epsilon \Delta N]^2]$$

[5] For the probability that two numbers, chosen at random, will both be prime, will be asymptotic to

$$\frac{\frac{\delta N_1}{\ln N_1}}{\delta N_1} \cdot \frac{\frac{\delta N_1}{\ln N_1} - 1}{\delta N_1 - 1} = \frac{1}{(\ln N_1)^2} \cdot \frac{\delta N_1}{\delta N_1 - 1}\left(1 - \frac{\ln N_1}{\delta N_1}\right) \sim \frac{1}{(\ln N_1)^2}$$

Hence, if A is less than unity, we will have

$$\epsilon AN_1(1 - A) \leq \delta N_1 \leq \epsilon AN_1,$$

so that the conditions stated at the head of this paragraph will be satisfied.

Hence the probability that any two numbers, chosen at random in such a subinterval of length $\epsilon\Delta N$, will both be prime, will be asymptotic to $\frac{1}{(\ln N_1)^2} \sim \frac{1}{(\ln N)^2}$, as N tends to infinity. But the probability that any two numbers, chosen at random in an interval of length $\epsilon\Delta N$ included in the larger interval $(N, N + \Delta N)$, will both be prime is just the probability that any two numbers, chosen at random in the interval $(N, N + \Delta N)$, but differing by less than $\epsilon\Delta N$, will both be prime. Since the latter probability is given by (31), we must have

$$\frac{\Delta N(\rho_1 + \rho_2 + \cdots + \rho_{[\epsilon\Delta N]})[1 + f(N, \epsilon)]}{\epsilon\Delta N\left\{\Delta N - \frac{1}{2}[\epsilon\Delta N] + \frac{1}{2}\right\}} \sim \frac{1}{(\ln N)^2}$$

Or,

$$\frac{1}{\epsilon\Delta N}(\rho_1 + \rho_2 + \cdots + \rho_{[\epsilon\Delta N]})\left\{\frac{1 + f(N, \epsilon)}{1 - \frac{[\epsilon\Delta N]}{2\Delta N} + \frac{1}{2\Delta N}}\right\} \sim \frac{1}{(\ln N)^2} \tag{32}$$

Now $\frac{[\epsilon\Delta N]}{2\Delta N} \leq \frac{\epsilon}{2}$, and if we take $N > \frac{1}{\epsilon A}$, then $\frac{1}{2\Delta N} = \frac{1}{2AN} < \frac{\epsilon}{2}$. Since $|f(N, \epsilon)| < \epsilon$, it follows that, for $N > \frac{1}{\epsilon A}$,

$$\frac{1 - \epsilon}{1 + \epsilon} < \frac{1 + f(N, \epsilon)}{1 - \frac{[\epsilon\Delta N]}{2\Delta N} + \frac{1}{2\Delta N}} < \frac{1 + \epsilon}{1 - \epsilon}.$$

Hence, from (32),

$$\begin{aligned} \overline{\lim_{N\to\infty}} \frac{(\ln N)^2}{\epsilon\Delta N}(\rho_1 + \rho_2 + \cdots + \rho_{[\epsilon\Delta N]}) &\leq \frac{1 + \epsilon}{1 - \epsilon} \\ \underline{\lim_{N\to\infty}} \frac{(\ln N)^2}{\epsilon\Delta N}(\rho_1 + \rho_2 + \cdots + \rho_{[\epsilon\Delta N]}) &\geq \frac{1 - \epsilon}{1 + \epsilon} \end{aligned} \tag{33}$$

Now suppose we let N tend to infinity through a certain sequence of values, defined by $N = \frac{M}{\epsilon A}$, where $M = 1, 2, 3, 4, \cdots$. Then as

$N \to \infty$, $M \to \infty$, and also $\epsilon\Delta N = \epsilon \cdot AN = M = [\epsilon \cdot AN]$. Hence

$$\overline{\lim_{M\to\infty}} \frac{\left(\ln \frac{M}{\epsilon A}\right)^2}{M} \sum_{k=1}^{M} \rho_k\left(\frac{M}{\epsilon A}\right) \leq \frac{1+\epsilon}{1-\epsilon}$$

$$\underline{\lim_{M\to\infty}} \frac{\left(\ln \frac{M}{\epsilon A}\right)^2}{M} \sum_{k=1}^{M} \rho_k\left(\frac{M}{\epsilon A}\right) \geq \frac{1-\epsilon}{1+\epsilon}$$

But, from (30), we may write

$$\overline{\lim_{M\to\infty}} \frac{\left(\ln \frac{M}{\epsilon A}\right)^2 \rho_2\left(\frac{M}{\epsilon A}\right)}{M} \sum_{k=1}^{M} \frac{\gamma_2^2}{\gamma_k^2} \cdot \Delta_{2k} \cdot \prod_{p=3}^{p\leq k}\left(1 + \frac{\Delta_{pk}}{p-2}\right) \leq \frac{1+\epsilon}{1-\epsilon} \tag{34}$$

and, from (25),

$$\rho_2\left(\frac{M}{\epsilon A}\right) \sim C_0 \cdot \frac{\Gamma^2}{\gamma_2^2} \cdot \frac{1}{\left(\ln \frac{M}{\epsilon A}\right)^2} \qquad (M \to \infty).$$

Hence (34) may be written

$$\overline{\lim_{M\to\infty}} C_0 \cdot \frac{\Gamma^2}{\gamma_2^2} \cdot \frac{1}{M} \cdot \sum_{k=1}^{M} \frac{\gamma_2^2}{\gamma_k^2} \cdot \Delta_{2k} \cdot \prod_{p=3}^{p\leq k}\left(1 + \frac{\Delta_{pk}}{p-2}\right) \leq \frac{1+\epsilon}{1-\epsilon}$$

and similarly,

$$\underline{\lim_{M\to\infty}} C_0 \cdot \frac{\Gamma^2}{\gamma_2^2} \cdot \frac{1}{M} \cdot \sum_{k=1}^{M} \frac{\gamma_2^2}{\gamma_k^2} \cdot \Delta_{2k} \cdot \prod_{p=3}^{p\leq k}\left(1 + \frac{\Delta_{pk}}{p-2}\right) \geq \frac{1-\epsilon}{1+\epsilon}$$

Letting ϵ tend to zero in the above limits, we obtain

$$\lim_{M\to\infty} \frac{C_0\Gamma^2}{\gamma_2^2} \cdot \frac{1}{M} \cdot \sum_{k=1}^{M} \frac{\gamma_2^2}{\gamma_k^2} \cdot \Delta_{2k} \cdot \prod_{p=3}^{p\leq k}\left(1 + \frac{\Delta_{pk}}{p-2}\right) = 1 \tag{35}$$

Now we have seen that, on the assumption that the exponent γ_2 is equal to Γ, we obtain $\rho_2(N) \sim \dfrac{C_0}{(\ln N)^2}$, and hence

$$\pi_3(N) \sim C_0 \cdot \int_2^N \frac{dt}{(\ln t)^2} \tag{36}$$

where

$$C_0 = 2 \cdot \prod_{p=3}^{\infty}\left[1 - \frac{1}{(p-1)^2}\right] = 1.3202\cdots \tag{37}$$

Now the distributions of all pairs of primes differing by any integer k are determined in a similar fashion; namely, by the superposition of two sieves for the determination of prime numbers, the second sieve being simply the first shifted k units to the left. Hence it seems reasonable to suppose that the distributions of any two sets of pairs of primes, differing by two integers k_1 and k_2, would be more closely related to each other than either would be related to the distribution of prime numbers. Hence it would seem more reasonable to make, instead of the assumption that $\gamma_2 = \Gamma$, the assumption that $\gamma_2 = \gamma_k$, for all values of k. We shall therefore make the latter assumption. We can then determine, from (35), the value of the constant factor $\frac{C_0\Gamma^2}{\gamma_2^2}$, and since, from (26),

$$\pi_2(N) \sim \frac{C_0\Gamma^2}{\gamma_2^2} \cdot \int_2^N \frac{dt}{(\ln t)^2} \tag{38}$$

we will then have another formula giving the approximate number of twin primes less than N.

We now proceed to determine the value of the constant $\frac{C_0\Gamma^2}{\gamma_2^2}$. Placing $\gamma_k = \gamma_2$ in (35), we obtain

$$\frac{\gamma_2^2}{C_0\Gamma^2} = \lim_{M\to\infty} \frac{1}{M} \sum_{k=1}^{M} \Delta_{2k} \prod_{p=3}^{p\leq k} \left(1 + \frac{\Delta_{pk}}{p-2}\right)$$

Letting $F(M) = \frac{1}{M} \cdot \sum_{k=1}^{M} \Delta_{2k} \cdot \prod_{p=3}^{p\leq k} \left(1 + \frac{\Delta_{pk}}{p-2}\right)$, this may be written

$$\frac{\gamma_2^2}{C_0\Gamma^2} = \lim_{M\to\infty} F(M) \tag{39}$$

Since $\Delta_{pk} = 0$ if $p > k$, we may replace $\prod_{p=3}^{p\leq k} \left(1 + \frac{\Delta_{pk}}{p-2}\right)$ by $\prod_{p=3}^{p\leq M} \left(1 + \frac{\Delta_{pk}}{p-2}\right)$, and expanding, obtain

$$F(M) = \frac{1}{M} \sum_{k=1}^{M} \Delta_{2k} \left\{1 + \sum_{p=3}^{p\leq M} \frac{\Delta_{pk}}{p-2} + \sum_{p=3}^{p\leq M} \sum_{p'=5}^{p'\leq M} \frac{\Delta_{pk}\Delta_{p'k}}{(p-2)(p'-2)} + \cdots\right\}$$

$$= \frac{1}{M} \left\{\sum_{k=1}^{M} \Delta_{2k} + \sum_{k=1}^{M} \sum_{p=3}^{p\leq M} \frac{\Delta_{2k}\Delta_{pk}}{p-2} + \sum_{k=1}^{M} \sum_{p=3}^{p<M} \sum_{p'=5}^{p'\leq M} \frac{\Delta_{2k}\Delta_{pk}\Delta_{p'k}}{(p-2)(p'-2)} + \cdots\right\}$$

where in the r-tuple summation over $p, p', p'', \cdots p^{(r-1)}$, we have $3 \leq p < p' < p'' < \cdots < p^{(r-1)} \leq M$. Since all terms are positive or zero, all methods of summation will be equivalent. Summing first on k, we obtain

$$F(M) = \frac{1}{M}\left\{\sum_{k=1}^{M} \Delta_{2k} + \sum_{p=3}^{p\leq M} \sum_{k=1}^{M} \frac{\Delta_{2k}\Delta_{pk}}{p-2} + \sum_{p=3}^{p\leq M} \sum_{p'=5}^{p'\leq M} \sum_{k=1}^{M} \frac{\Delta_{2k}\Delta_{pk}\Delta_{p'k}}{(p-2)(p'-2)} + \cdots\right\}$$

Now if $p_1, p_2, p_3, \cdots p_n$ are distinct primes, then, since $\Delta_{p_i k} = 0$ unless k is a multiple of p_i (in which event $\Delta_{p_i k} = 1$), it follows that $\Delta_{p_1 k} \cdot \Delta_{p_2 k} \cdot \Delta_{p_3 k} \cdot \cdots \cdot \Delta_{p_n k} = 0$ unless k is a multiple of $p_1 p_2 p_3 \cdots p_n$.

Hence $\sum_{k=1}^{M} (\Delta_{p_1 k}\Delta_{p_2 k}\Delta_{p_3 k} \cdots \Delta_{p_n k}) = \left[\frac{M}{p_1 p_2 p_3 \cdots p_n}\right]$, where the notation $[x]$ means the greatest integer less than or equal to x. We may therefore write

$$F(M) = \frac{1}{M}\left\{\left[\frac{M}{2}\right] + \sum_{p=3}^{p\leq M} \frac{1}{p-2} \cdot \left[\frac{M}{2p}\right] + \sum_{p=3}^{p\leq M} \sum_{p'=5}^{p'\leq M} \frac{1}{(p-2)(p'-2)} \cdot \left[\frac{M}{2pp'}\right] + \cdots\right\}$$

where in the r-tuple summation over $p, p', p'', \cdots p^{(r-1)}$, we have $3 \leq p < p' < p'' < \cdots < p^{(r-1)} \leq M$.

Hence if we let

$$\phi(M) = \frac{1}{M}\left\{\frac{M}{2} + \sum_{p=3}^{p\leq M} \frac{1}{p-2} \cdot \frac{M}{2p} + \sum_{p=3}^{p\leq M} \sum_{p'=5}^{p'\leq M} \frac{1}{(p-2)(p'-2)} \cdot \frac{M}{2pp'} + \cdots\right\} \tag{40}$$

it follows that $\phi(M) - F(M) \geq 0$, and

$$\phi(M) - F(M) < \frac{1}{M}\left\{1 + \sum_{p=3}^{p\leq M} \frac{1}{p-2} + \sum_{p=3}^{p\leq M} \sum_{p'=5}^{p'\leq M} \frac{1}{(p-2)(p'-2)} + \cdots\right\}$$

where $3 \leq p < p' < p'' < \cdots < p^{(r-1)} \leq M$. But the expression on the right is precisely equal to the expansion of the product $\prod_{p=3}^{p \leq M} \left[1 + \frac{1}{(p-2)}\right]$, and hence

$$0 \leq \phi(M) - F(M) < \frac{1}{M} \prod_{p=3}^{p \leq M} \left(1 + \frac{1}{p-2}\right)$$

$$= \frac{1}{M} \prod_{p=3}^{p \leq M} \left(\frac{p}{p-1}\right) \cdot \prod_{p=3}^{p \leq M} \left[1 + \frac{1}{p(p-2)}\right]$$

But the product $\prod_{p=3}^{p \leq M} \left[1 + \frac{1}{p(p-2)}\right]$ obviously converges, for M becoming infinite, to a certain constant value K. Since this product is also a monotonic increasing function of M, it follows that

$$0 \leq \phi(M) - F(M) < \frac{K}{M} \cdot \prod_{p=3}^{p \leq M} \left(\frac{p}{p-1}\right)$$

Now it is known[6] that

$$\prod_{p=2}^{p \leq M} \left(\frac{p-1}{p}\right) \sim \frac{e^{-c}}{\ln M}$$

where c is Euler's constant. (This is equivalent to (9).) Hence,

$$0 \leq \phi(M) - F(M) < \frac{K}{M} \prod_{p=3}^{p \leq M} \left(\frac{p}{p-1}\right) \sim \frac{Ke^{c}}{2} \cdot \frac{\ln M}{M}$$

whence it is obvious that $\phi(M) - F(M) \to 0$ as $M \to \infty$.

Now, from (40),

$$\phi(M) = \frac{1}{2}\left\{1 + \sum_{p=3}^{p \leq M} \frac{1}{p(p-2)} + \sum_{p=3}^{p \leq M} \sum_{p'=4}^{p' \leq M} \frac{1}{pp'(p-2)(p'-2)} + \cdots\right\}$$

$$= \frac{1}{2} \cdot \prod_{p=3}^{p \leq M} \left[1 + \frac{1}{p(p-2)}\right],$$

and, from (39),

$$\frac{\gamma_3^2}{C_0 \Gamma^2} = \lim_{M \to \infty} F(M) = \lim_{M \to \infty} \{\phi(M) - [\phi(M) - F(M)]\}.$$

[6] Edmund Landau, "Handbuch der Lehre von der Verteilung der Primzahlen", §36, p. 140.

But we have shown that $\lim_{M\to\infty}[\phi(M) - F(M)] = 0$, and hence

$$\frac{\gamma_2^2}{C_0\Gamma^2} = \lim_{M\to\infty}\phi(M) = \frac{1}{2}\cdot\prod_{p=3}^{\infty}\left[\frac{(p-1)^2}{p(p-2)}\right]$$

But, from (37),

$$\frac{1}{C_0} = \frac{1}{2\cdot\prod_{p=3}^{\infty}\left[1 - \frac{1}{(p-1)^2}\right]} = \frac{1}{2}\cdot\prod_{p=3}^{\infty}\left[\frac{(p-1)^2}{p(p-2)}\right]$$

so that we obtain, finally, the surprising result

$$\frac{\gamma_2^2}{C_0\Gamma^2} = \frac{1}{C_0}$$

or,

$$\gamma_2 = \Gamma \tag{41}$$

Thus we see from (38) that the assumption that $\gamma_k = \gamma_2$ for all values of k greater than zero leads to the same formula for the distribution of twin primes as the assumption $\gamma_2 = \Gamma$. We may interpret this result as follows:

Let $\Delta_{2k}\cdot\prod_{p=3}^{p\le k}\left(1 + \frac{\Delta_{pk}}{p-2}\right) = w(k)$. Then (35) may be written

$$\lim_{M\to\infty}\frac{1}{M}\cdot\sum_{k=1}^{M}\frac{1}{\gamma_k^2}\cdot w(k) = \frac{1}{C_0\Gamma^2}$$

But we have just shown that

$$\lim_{M\to\infty}\frac{1}{M}\sum_{k=1}^{M}w(k) = \frac{1}{C_0}$$

and hence, by division,

$$\frac{1}{\Gamma^2} = \lim_{M\to\infty}\frac{\sum_{k=1}^{M}\frac{1}{\gamma_k^2}w(k)}{\sum_{k=1}^{M}w(k)} \tag{42}$$

Thus we see that the reciprocal square of the exponent for prime numbers may be expressed as a weighted mean of the reciprocal squares of the exponents for all pairs of primes differing by integers. The

unique way in which Δ_{2k} enters into the weight function $w(k)$ is only apparent, and may be removed by dividing $w(k)$ by a constant factor. If α is any even number, then $w(\alpha)$ will not vanish, so that we may replace $w(k)$ by a new weight function

$$W(k) = \frac{w(k)}{w(\alpha)} = \frac{\Delta_{2k}}{\Delta_{2\alpha}} \cdot \prod_{p=3}^{\infty}\left(\frac{p-2+\Delta_{pk}}{p-2+\Delta_{p\alpha}}\right) = \prod_{p=3}^{\infty}\left(\frac{p-2+\Delta_{pk}}{p-2+\Delta_{p\alpha}}\right)$$

which is seen to be perfectly symmetrical.

It is obvious from (42) why the assumption that the exponents γ_k are all equal results in the conclusion that their common value is Γ. Of course, if these exponents are not all equal, we should not expect γ_2 to be equal to Γ, and conversely, if γ_2 is not equal to Γ, then the exponents can certainly not all be equal. Unfortunately, the very rough and approximate line of reasoning which we have pursued does not permit us to draw any very strong conclusions regarding the exact value of γ_2. We can only say that, since the distributions of pairs of primes differing by any integer k are determined in more or less similar fashions (by the superposition of two sieves for the determination of prime numbers), and since a certain weighted average of the exponents γ_k is equal to Γ, we should not expect the exponents γ_k to differ very much from Γ. In view of the relation (42), not knowing anything very definite about the relative magnitudes of the exponents γ_k, the best guess we can make concerning the value of γ_2 is that $\gamma_2 = \Gamma$. We should therefore expect the formula (36) to give at least a very rough approximation to the number of twin primes less than a given limit.

V. Comparison of Values Given by $f(N) = C_0 \cdot \int_2^N \frac{dt}{(\ln t)^2}$ with Count of Twin Primes up to 800,000

In order to check the values given by the formula (36), it is first necessary to compute the constant C_0. The infinite product (37) does not converge very rapidly, but it is possible, by using the infinite product expansion for the Riemann zeta-function, to obtain more rapidly convergent products. As $p \to \infty$, $\frac{1}{(p-1)^2} \sim \frac{1}{p^2}$, and hence if we multiply by the infinite product $\prod_{p=3}^{\infty}\left(1+\frac{1}{p^2}\right)$, we shall obtain a more rapidly convergent product.

Now,

$$\frac{\zeta(2)}{\zeta(4)} = \frac{15}{\pi^2} = \frac{\prod_{p=2}^{\infty}\left(1-\frac{1}{p^4}\right)}{\prod_{p=2}^{\infty}\left(1-\frac{1}{p^2}\right)} = \frac{5}{4}\cdot\prod_{p=3}^{\infty}\left(1+\frac{1}{p^2}\right)$$

and hence

$$\frac{C_0}{2}\cdot\frac{12}{\pi^2} = \prod_{p=3}^{\infty}\frac{p^2-2p}{(p-1)^2}\cdot\frac{p^2+1}{p^2} = \prod_{p=3}^{\infty}\frac{p^4-2p^3+p^2-2p}{p^4-2p^3+p^2}$$

$$\therefore \quad C_0 = \frac{\pi^2}{6}\cdot\prod_{p=3}^{\infty}\left[1-\frac{2}{p(p-1)^2}\right] \tag{43}$$

This product converges fairly rapidly and may readily be used for computation.[7]

In order to estimate the error involved in cutting off the infinite product (43) after a finite number of factors, let

$$R_N = \prod_{p\geq N}\left[1-\frac{2}{p(p-1)^2}\right].$$

Then,

$$\frac{1}{R_N} = \prod_{p\geq N}\frac{p^3-2p^2+p}{p^3-2p^2+p-2} = \prod_{p\geq N}\left[1+\frac{2}{(p-2)(p^2+1)}\right]$$

$$-\ln R_N = \sum_{p\geq N}\ln\left[1+\frac{2}{(p-2)(p^2+1)}\right]$$

$$\therefore \quad 0 < -\ln R_N < \sum_{p\geq N}\frac{2}{(p-2)(p^2+1)} < \sum_{n=N-2}^{\infty}\frac{2}{n^3} < \int_{N-3}^{\infty}\frac{2dt}{t^3}$$

Therefore

$$0 < -\ln R_N < \frac{1}{(N-3)^2}.$$

Hence it is seen that if we calculate the product $\frac{\pi^2}{6}\cdot\prod_{p<100}\left[1-\frac{2}{p(p-1)^2}\right]$, the error involved in using this as the value of C_0 must be less than

[7] It is obviously possible to continue this process. Thus we readily obtain

$$C_0 = \frac{\pi^8}{4410}\cdot\frac{1}{[\zeta(3)]^2}\cdot\prod_{p=3}^{\infty}\left[1-\frac{2(2p^2-p+1)}{p(p-1)^2(p^3-1)}\right]$$

which converges still more rapidly, but the terms in the product are more trouble to calculate.

0.00016. Since the density of primes for $N > 100$ will actually be considerably less than the density of integers (for $N = 100$, the density is about $\frac{1}{\ln 100} = .22$), the error cannot be much greater than (.22) (.00016) = .000035.

Calculating the product $\frac{\pi^2}{6} \prod_{p=3}^{97} \left[1 - \frac{2}{p(p-1)^2}\right]$, we find

$$C_0 = 1.3202 \tag{44}$$

which is in all probability correct to four decimal places, and is certainly not in error by more than 0.00022.

Now,

$$\int_2^N \frac{dt}{(\ln t)^2} = \int_2^N \frac{dt}{\ln t} - \frac{N}{\ln N} + \frac{2}{\ln 2},$$

and since there exist several tables[8] giving values of the logarithmic integral function $Li(x) = \int_0^x \frac{dt}{\ln t}$, it is readily possible to compute values of the function $f(N) = C_0 \cdot \int_2^N \frac{dt}{(\ln t)^2}$.

Using Lehmer's "Table of Prime Numbers", a count of twin primes up to 800,000 has been made. We give now a brief table comparing values of $f(N)$ with the actual number of twin primes less than N, represented by $\pi_t(N)$. (A more complete tabulation of $\pi_t(N)$ is given in the appendix.)

It is thus seen that $f(N)$ gives a fairly good approximation to $\pi_t(N)$, but is in error by slightly over 1% for the values of N greater than 200,000. This would seem to indicate that the constant factor C_0 is slightly too large. However, it is also seen that for smaller values of N, the percent deviation becomes considerably greater than 1%, remaining of constant sign. Hence, even if the factor C_0 were replaced by one slightly smaller, the new formula thus obtained would still give values definitely too large for small values of N. The reason for the latter observation is easy to see, however.

A glance at Table II, at the end of Chapter I, will show that von Mangoldt's logarithmic integral formula gives values which are definitely too large, for small values of N, though the percent deviation tends to zero as N increases. Riemann's formula gives a much closer approxima-

[8] E. Jahnke—F. Emde, "Funktionentafeln mit Formeln und Kurven", B. G. Teubner, Leipzig, 1933.

tion for all values of N up to 1,000,000,000. Now our general theory indicates that $\pi_1(N)$ should be approximately represented by a certain constant times the integral of the square of the derivative of a function giving the average density of primes. Hence we should obtain a better approximation to $\pi_1(N)$ if we take as the average density of primes,

TABLE IV

N	$\pi_1(N)$	$f(N)$	Diff.
100	9	13	+4
200	16	19	+3
400	22	27	+5
600	28	35	+7
800	31	40	+9
1 000	36	46	+10
2 000	62	70	+8
4 000	104	112	+8
6 000	144	148	+4
8 000	176	182	+6
10 000	206	214	+8
20 000	343	357	+14
40 000	592	607	+15
60 000	811	833	+22
80 000	1006	1045	+39
100 000	1222	1248	+26
200 000	2157	2182	+25
300 000	2991	3037	+46
400 000	3801	3848	+47
500 000	4562	4628	+66
600 000	5328	5384	+56
700 000	6058	6121	+63
800 000	6763	6842	+79

instead of $\frac{1}{\ln N}$, the derivative of Riemann's function $R(N)$. We shall therefore now consider the function

$$F(N) = C \cdot \int_2^N [R'(t)]^2 \, dt \qquad (45)$$

where $R'(N)$ is the derivative of Riemann's function, and C is a constant $\left(\text{equal to } \frac{C_0 \Gamma^2}{\gamma_2^2}\right)$.

VI. The Analogue of Riemann's Formula for Twin Primes

Now Riemann's function $R(x)$ is defined by

$$R(x) = A + \sum_{n=1}^{\infty} \frac{\mu(n)}{n} Li(x^{1/n}) \tag{46}$$

where $Li(x) = \int_0^x \frac{dt}{\ln t}$. Hence,

$$R'(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \cdot \frac{d}{dx} \int_0^{x^{1/n}} \frac{dt}{\ln t} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \cdot \frac{1}{n} x^{(1/n)-1} \cdot \frac{1}{\ln (x^{1/n})}$$

or,

$$R'(x) = \frac{1}{x \ln x} \cdot \sum_{n=1}^{\infty} \frac{\mu(n)}{n} x^{1/n} = \frac{1}{x \ln x} \left\{ x + \sum_{n=2}^{\infty} \frac{\mu(n)}{n} x^{1/n} \right\}$$

If we let

$$\phi(x) = \sum_{n=2}^{\infty} \frac{\mu(n)}{n} x^{1/n} \tag{47}$$

then,

$$R'(x) = \frac{1}{x \ln x} [x + \phi(x)] \tag{48}$$

From (47),

$$\phi(x) = \sum_{n=2}^{\infty} \frac{\mu(n)}{n} \cdot e^{1/n \ln x} = \sum_{n=2}^{\infty} \frac{\mu(n)}{n} \sum_{m=0}^{\infty} \frac{(\ln x)^m}{n^m \cdot m!}$$

$$= \sum_{n=2}^{\infty} \frac{\mu(n)}{n} \left[1 + \sum_{m=1}^{\infty} \frac{(\ln x)^m}{n^m \cdot m!} \right]$$

Now it is well known that $\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0$, and hence $\sum_{n=2}^{\infty} \frac{\mu(n)}{n} = -1$. Therefore

$$\phi(x) + 1 = \sum_{n=2}^{\infty} \frac{\mu(n)}{n} \sum_{m=1}^{\infty} \frac{(\ln x)^m}{n^m \cdot m!}$$

If it is permissible to invert the order of summation, we have

$$\phi(x) + 1 = \sum_{m=1}^{\infty} \frac{(\ln x)^m}{m!} \sum_{n=2}^{\infty} \frac{\mu(n)}{n^{m+1}},$$

and since $\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = 1 + \sum_{n=2}^{\infty} \frac{\mu(n)}{n^s}$, for $s > 1$,

$$\phi(x) + 1 = \sum_{m=1}^{\infty} \frac{(\ln x)^m}{m!} \left[\frac{1 - \zeta(m+1)}{\zeta(m+1)} \right] \tag{49}$$

But this series is absolutely convergent, and hence the above inversion of order is justified.

Now, for $m \geq 1$,

$$0 < \zeta(m+1) - 1 = \frac{1}{2^{m+1}} + \sum_{n=3}^{\infty} \frac{1}{n^{m+1}} < \frac{1}{2^{m+1}} + \int_2^{\infty} \frac{dt}{t^{m+1}}$$

$$0 < \zeta(m+1) - 1 < \frac{1}{2^{m+1}} + \frac{1}{m \cdot 2^m} = \frac{1}{2^m}\left(\frac{1}{2} + \frac{1}{m}\right) \leq \frac{1.5}{2^m}$$

Since $1 < \zeta(m+1) < 2$, for $m \geq 1$, it follows that

$$0 < \frac{\zeta(m+1) - 1}{\zeta(m+1)} < \frac{1.5}{2^m}$$

Hence, from (49), for $x \geq 1$,

$$0 < -[\phi(x) + 1] < 1.5 \sum_{m=1}^{\infty} \frac{(\ln x)^m}{2^m \cdot m!} = 1.5(\sqrt{x} - 1)$$

or,
$$1 < -\phi(x) < 1.5\sqrt{x} - 0.5 < 1.5\sqrt{x}$$

Squaring, we obtain

$$1 < [\phi(x)]^2 < \frac{9}{4}x \qquad (x \geq 1) \tag{50}$$

Now, from (48),

$$\int_2^N [R'(x)]^2\,dx = \int_2^N \frac{dx}{(\ln x)^2} + \int_2^N \frac{2\phi(x)\,dx}{x(\ln x)^2} + \int_2^N \frac{[\phi(x)]^2\,dx}{x^2(\ln x)^2} \tag{51}$$

and from (50), for $N > 2$,

$$0 < \int_2^N \frac{[\phi(x)]^2\,dx}{x^2(\ln x)^2} < \frac{9}{4} \cdot \int_2^N \frac{dx}{x(\ln x)^2} = -\frac{9}{4} \cdot \left[\frac{1}{\ln x}\right]_2^N = \frac{9}{4} \cdot \left[\frac{1}{\ln 2} - \frac{1}{\ln N}\right]$$

Hence, for $N > 2$, this integral is less than $\dfrac{9}{4 \ln 2} = 3.25$, and so may be neglected in (51) without appreciably changing the value of $\int_2^N [R'(x)]^2\,dx$. Hence, substituting the value of $\phi(x)$ as given in (47), we may write (51) as

$$\int_2^N [R'(x)]^2\,dx = \int_2^N \frac{dx}{(\ln x)^2} + 2\sum_{n=2}^{\infty} \frac{\mu(n)}{n} \int_2^N \frac{x^{(1/n)-1}\,dx}{(\ln x)^2} + O(1) \tag{52}$$

where the term represented by $O(1)$ is less than 3.25 for $N > 2$.

Now place $x^{(1/n)-1}dx = n\,dt$; $x^{1/n} = t$; $\ln x = n \ln t$; then

$$\int_2^N \frac{x^{(1/n)-1}dx}{(\ln x)^2} = \frac{1}{n}\cdot\int_{2^{1/n}}^{N^{1/n}} \frac{dt}{(\ln t)^2} = \frac{1}{n}\cdot\int_2^{N^{1/n}} \frac{dt}{(\ln t)^2} + \frac{1}{n}\cdot\int_{2^{1/n}}^2 \frac{dt}{(\ln t)^2}$$

and hence

$$\sum_{n=2}^{\infty} \frac{\mu(n)}{n}\int_2^N \frac{x^{(1/n)-1}dx}{(\ln x)^2} = \sum_{n=2}^{\infty} \frac{\mu(n)}{n^2}\int_2^{N^{1/n}} \frac{dt}{(\ln t)^2} + \sum_{n=2}^{\infty} \frac{\mu(n)}{n^2}\int_{2^{1/n}}^2 \frac{dt}{(\ln t)^2} \tag{53}$$

It can readily be shown that the second summation on the right converges to a constant value. Placing $\ln t = x$, $dt = e^x dx$, we have

$$\begin{aligned} \int_{2^{1/n}}^2 \frac{dt}{(\ln t)^2} &= \int_{1/n \ln 2}^{\ln 2} \frac{e^x}{x^2}dx = \int_{1/n \ln 2}^{\ln 2} \left\{\frac{1}{x^2} + \frac{1}{x} + \frac{e^x - x - 1}{x^2}\right\}dx \\ \int_{2^{1/n}}^2 \frac{dt}{(\ln t)^2} &= \frac{n}{\ln 2} - \frac{1}{\ln 2} + \ln n + \int_{1/n \ln 2}^{\ln 2} \frac{e^x - x - 1}{x^2}dx \end{aligned} \tag{54}$$

Now, for $0 \le x \le \ln 2$, $0 < \dfrac{e^x - x - 1}{x^2} < 1$, and hence

$$0 < \int_{1/n \ln 2}^{\ln 2} \frac{e^x - x - 1}{x^2}\,dx < \ln 2 \tag{55}$$

From (54)

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{\mu(n)}{n^2}\int_{2^{1/n}}^2 \frac{dt}{(\ln t)^2} = \sum_{n=2}^{\infty} \frac{\mu(n)}{n^2}\Bigg\{ &\frac{n}{\ln 2} - \frac{1}{\ln 2} \\ &+ \ln n + \int_{1/n \ln 2}^{\ln 2} \frac{e^x - x - 1}{x^2}dx\Bigg\} \end{aligned} \tag{56}$$

But from (55),

$$\begin{aligned} \left|\sum_{n=2}^{\infty} \frac{\mu(n)}{n^2}\int_{1/n \ln 2}^{\ln 2} \frac{e^x - x - 1}{x^2}dx\right| &< \sum_{n=2}^{\infty}\left|\frac{\mu(n)}{n^2}\int_{1/n \ln 2}^{\ln 2} \frac{e^x - x - 1}{x^2}dx\right| \\ &< \sum_{n=2}^{\infty} \frac{\ln 2}{n^2} < \frac{1}{2} \end{aligned}$$

Also,

$$\sum_{n=2}^{\infty}\left[-\frac{1}{\ln 2}\cdot\frac{\mu(n)}{n^2}\right] = -\frac{1}{\ln 2}\left[\frac{1}{\zeta(2)} - 1\right] = 0.57\cdots$$

$$\left|\sum_{n=2}^{\infty} \frac{\mu(n)\ln n}{n^2}\right| = \left|\frac{\zeta'(2)}{[\zeta(2)]^2}\right| < \frac{1}{2}$$

$$\sum_{n=2}^{\infty} \frac{\mu(n)}{n \ln 2} = -\frac{1}{\ln 2} = -1.45\cdots$$

The above relations suffice to show that the series (56) converges, and that its sum will not be greater than 2 in absolute value. Hence we may write (53) as

$$\sum_{n=2}^{\infty} \frac{\mu(n)}{n} \int_2^N \frac{x^{(1/n)-1}\,dx}{(\ln x)^2} = \sum_{n=2}^{\infty} \frac{\mu(n)}{n^2} \int_2^{N^{1/n}} \frac{dt}{(\ln t)^2} + O(1) \qquad (57)$$

where the term represented by $O(1)$ is less than 2 in absolute value.

Since the term $O(1)$ occurring in (52) is less than 3.25 in absolute value, we may substitute (57) in (52), obtaining finally

$$\int_2^N [R'(x)]^2\,dx = \int_2^N \frac{dx}{(\ln x)^2} + 2 \cdot \sum_{n=2}^{\infty} \frac{\mu(n)}{n^2} \int_2^{N^{1/n}} \frac{dt}{(\ln t)^2} + O(1)$$

where the term $O(1)$ is less than 7.25 in absolute value. If we let $\theta(x) = \int_2^x \frac{dt}{(\ln t)^2}$, the above equation may be written in the form

$$\int_2^N [R'(x)]^2\,d\theta x = \theta(N) + 2 \cdot \sum_{n=2}^{\infty} \frac{\mu(n)}{n^2} \theta(N^{1/n}) + O(1) \qquad (58)$$

and the analogy with (46) is immediately evident.

Neglecting the bounded term $O(1)$, we shall write, in accordance with (45)

$$F(N) = C \cdot \left\{ \theta(N) + 2 \cdot \sum_{n=2}^{\infty} \frac{\mu(n)}{n^2} \theta(N^{1/n}) \right\} \qquad (59)$$

and see if it is possible to choose the constant C so that a good approximation to $\pi_i(N)$ is obtained. Because of the rapidity with which the series (59) converges, it suffices, in order to obtain values of $F(N)$ correct to the nearest integer, to take only the first five terms of the series (59). Since

$$\theta(N) = \int_2^N \frac{dt}{(\ln t)^2} = \int_2^N \frac{dt}{\ln t} - \frac{N}{\ln N} + \frac{2}{\ln 2},$$

values of $\theta(N)$ may readily be computed from tables of the logarithmic integral function.

We give now a table comparing values of $F(N)$ with the corresponding values of $\pi_i(N)$. Without attempting to determine the best possible value of the constant C, we have taken C equal to 1.310, and it is readily seen that a surprisingly good approximation to the function $\pi_i(N)$ is obtained.

TABLE V

N	$\pi_t(N)$	$F(N)$	Diff.
100	9	10	+1
200	16	15	+1
400	22	23	+1
600	28	29	+1
800	31	35	+4
1 000	36	40	+4
2 000	62	64	+2
4 000	104	104	0
6 000	144	140	−4
8 000	176	173	−3
10 000	206	204	−2
20 000	343	345	+2
40 000	592	591	−1
60 000	811	814	+3
80 000	1006	1024	+18
100 000	1222	1225	+3
150 000	1698	1700	+2
200 000	2157	2148	−9
250 000	2585	2578	−7
300 000	2991	2995	+4
350 000	3419	3401	−18
400 000	3801	3798	−3
450 000	4191	4187	−4
500 000	4562	4570	+8
550 000	4944	4947	+3
600 000	5328	5319	−9
650 000	5706	5687	−19
700 000	6058	6050	−8
750 000	6411	6409	−2
800 000	6763	6765	+2

VII. Conclusions

We have just shown that the function defined by

$$F(N) = C \cdot \int_2^N [R'(t)]^2 \, dt,$$

where $C = 1.310$, gives a very good approximation to $\pi_t(N)$ for values of N less than 800,000. Insofar as the theory developed in chapters III and IV leads us to expect the existence of a relation of the form

$$\pi_t(N) \sim (\text{Const.}) \int_2^N [\rho(t)]^2 \, dt,$$

and since this supposition is well checked by actual count of twin primes up to 800,000, it seems extremely probable that a relation of the above form actually does hold, as $N \to \infty$. Since

$$\int_2^N [R'(t)]^2\,dt \sim \int_2^N \frac{dt}{(\ln t)^2}$$

(this follows immediately from (48) and (50)), it therefore seems probable that, as $N \to \infty$,

$$\pi_t(N) \sim K \cdot \int_2^N [R'(t)]^2\,dt \sim K \cdot \int_2^N \frac{dt}{(\ln t)^2} \sim \frac{KN}{(\ln N)^2} \qquad (60)$$

where the constant K does not differ very much from 1.310. The exact determination of this constant—assuming it exists—should furnish an interesting problem.

We may also include that the original simple assumption that $\gamma_2 = \Gamma$ is false, and that instead, we have $\frac{C_0\Gamma^2}{\gamma_2^2} = C$, or $\gamma_2^2 = \frac{C_0}{C}\Gamma^2$, whence $\gamma_2 = (1.0004)\Gamma = 0.5637$, approximately.[9]

Appendix

We give now a tabulation of the function $\pi_t(N)$, for values of N up to 800,000. The first part of this tabulation, which follows immediately, gives values of $\pi_t(N)$ at intervals of 1000 up to 100,000. In the left hand side of each double column is given the value of N divided by 1,000, and opposite is given the corresponding value of $\pi_t(N)$.

1	36	35	540	69	892
2	62	36	551	70	904
3	83	37	560	71	920
4	104	38	572	72	930
5	127	39	583	73	941
6	144	40	592	74	947
7	163	41	600	75	957
8	176	42	612	76	966

[9] The possibility must not be overlooked, however, that a certain constant error of omission in the counting of the twin primes has been committed. The author hopes, at some future time, to recheck the count of twin primes, and continue the count to a higher limit. The omission of about six or seven twin primes per hundred thousand integers would make the empirically determined constant C equal to C_0, but an error as large as this is not probable.

9	191	43	624	77	977
10	206	44	635	78	986
11	222	45	646	79	995
12	236	46	653	80	1006
13	247	47	665	81	1019
14	262	48	675	82	1031
15	273	49	688	83	1043
16	285	50	705	84	1051
17	298	51	714	85	1063
18	316	52	728	86	1074
19	328	53	736	87	1082
20	343	54	747	88	1096
21	358	55	755	89	1106
22	373	56	766	90	1114
23	389	57	780	91	1124
24	403	58	789	92	1134
25	409	59	800	93	1146
26	421	60	811	94	1158
27	432	61	821	95	1171
28	447	62	827	96	1179
29	459	63	835	97	1186
30	468	64	845	98	1201
31	479	65	854	99	1214
32	492	66	868	100	1222
33	511	67	874		
34	524	68	883		

There now follows a tabulation of values of $\pi_1(N)$ at intervals of 10,000 up to 800,000. In the left hand side of each double column is given the value of N divided by 10,000.

1	206	30	2991	59	5253
2	343	31	3080	60	5328
3	468	32	3167	61	5407
4	592	33	3270	62	5476
5	705	34	3342	63	5552
6	811	35	3419	64	5628
7	904	36	3499	65	5706
8	1006	37	3571	66	5774
9	1114	38	3640	67	5858
10	1222	39	3718	68	5928

11	1318	40	3801	69	5990
12	1422	41	3878	70	6058
13	1514	42	3958	71	6133
14	1605	43	4051	72	6198
15	1698	44	4116	73	6272
16	1798	45	4191	74	6339
17	1879	46	4268	75	6411
18	1974	47	4333	76	6474
19	2072	48	4414	77	6539
20	2157	49	4484	78	6615
21	2250	50	4562	79	6687
22	2326	51	4629	80	6763
23	2419	52	4709		
24	2500	53	4800		
25	2585	54	4873		
26	2660	55	4944		
27	2750	56	5015		
28	2832	57	5088		
29	2918	58	5175		

REMARK ON THE THEORY OF BENDING OF PLATES OF VARIABLE THICKNESS

By M. Eric Reissner*

The purpose of the present paper is to deduce in a simpler way a solution given by R. Gran Olsson[1] of the differential equation of plates of linearly varying stiffness using a transformation of this equation stated by the author in a previous paper.[2]

The equation in question is[3]

$$\nabla^2(N\nabla^2 w) + (1-\nu)\left(2\frac{\partial^2 N}{\partial x\,\partial y}\frac{\partial^2 w}{\partial x\,\partial y} - \frac{\partial^2 N}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 N}{\partial y^2}\frac{\partial^2 w}{\partial x^2}\right) = p \quad (1)$$

in which
w denotes the deflection,
N the stiffness,
x, y the coordinates in the plane of the plate,
ν Poisson's Number and
p the acting load.

For linearly varying stiffness

$$N = cx \quad (2)$$

equation (1) reduces to

$$\nabla^2(x\nabla^2 w) = p \quad (3)$$

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[1] *R. Gran Olsson*, Proc. 8th Skandinavian Congress of Math., Stockholm 1934, p. 149. Ingenieur-Archiv. 5 (1934), p. 363.

[2] E. Reissner, Ingenieur-Archiv. 7 (1936), p. 82.

[3] It is obtained by introducing into the condition of equilibrium of an element of the plate, that is into:

$$\frac{\partial^2 Mx}{\partial x^2} + 2\frac{\partial H}{\partial x\partial y} + \frac{\partial^2 My}{\partial y^2} = -p(x, y)$$

the stress-strain relations

$$M_x = -N(x, y)\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right);\ M_y = -N\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right);\ H = -N(1-\nu)\frac{\partial^2 w}{\partial x\partial y}$$

Under the supposition that two opposite edges say $y = 0$ and $y = b$ are simply supported, we can solve the homogeneous equation

$$p = 0$$

by assuming in the usual way

$$w = \sin \beta y \cdot f(x) \tag{4}$$

which for the plate of constant thickness is due to Maurice Lévy[4] and get from (3)

$$\left[\left(\frac{d^2}{dx^2} - \beta^2\right)x\left(\frac{d^2}{dx^2} - \beta^2\right)\right]f(x) = 0 \tag{5}$$

It follows at once

$$\left(\frac{d^2}{dx^2} - \beta^2\right)f(x) = \frac{ae^{\beta x} + be^{-\beta x}}{x} \tag{6}$$

in which a and b are arbitrary constants. The integral of (6) is

$$f(x) = Ce^{\beta x} + De^{-\beta x} + f_0(x) \tag{7}$$

and f_0 is obtained by using Lagrange's method of variation of constants. Assuming

$$f_0 = c(x)e^{\beta x} + d(x)e^{-\beta x} \tag{8}$$

we satisfy (6) if we put

$$\left.\begin{aligned} c'e^{\beta x} + d'e^{-\beta x} &= 0 \\ c'e^{\beta x} - d'e^{-\beta x} &= \frac{ae^{\beta x} + be^{-\beta x}}{\beta x} \end{aligned}\right\} \tag{9}$$

This yields

$$\left.\begin{aligned} c &= \int\left(\frac{a}{2\beta x} - \frac{be^{-2\beta x}}{2\beta x}\right)dx \\ d &= -\int\left(\frac{b}{2\beta x} + \frac{ae^{+2\beta x}}{2\beta x}\right)dx \end{aligned}\right\} \tag{10}$$

(8) and (10) give

$$f_0 = A[ln\, 2\beta x - e^{-2\beta x}Ei(2\beta x)]e^{\beta x} - B[ln\, 2\beta x - e^{2\beta x}Ei(-2\beta x)]e^{-\beta x} \tag{11}$$

[4] M. Lévy, Comptes-Rendues t. 129 (1899).

where A and B are arbitrary constants and

$$Ei(x) = \int_{-\infty}^{x} \frac{e^u}{u}\, du$$

is the so called "Exponential Integral", which is a tabulated function. Combining (7) and (11) we get the required solution of (3) and with it the surface deflection and the stresses and strains of rectangular plates of stiffness being a linear function along two of the edges, simply supported on these edges, carrying no surface load but arbitrary edge forces and couples on the two other edges. The solution (7) and (11) corresponds exactly to the solution in Gran Olsson's papers quoted under[1] and might easily be extended for any arbitrary load function.

THE MAPPING OF GRAPHS ON SURFACES*

By I. N. Kagno†

Introduction. Kuratowski has shown that a necessary and sufficient condition that a finite graph cannot be mapped on a plane is that it contains as subgraph one of the two graphs; A: the graph formed by joining each vertex of the triad (a, b, c) to each vertex of the triad (r, s, t). B: the graph formed by joining each vertex of the set (a, b, c, d, e) to each of the others.[1] A purely combinatorial proof of the same theorem was given by Whitney.[2]

Kuratowski states that these graphs can be mapped on the projective plane and the torus. In this paper we study the nature of the maps of A and B on these surfaces. We show how to construct irreducible[3] graphs which cannot be mapped on a projective plane, and irreducible graphs which cannot be mapped on a torus. We also show how to construct irreducible non Σ_p-al, and irreducible non $\mathfrak{S}_q$-al graphs, where Σ_p is an orientable surface of genus p, and $\mathfrak{S}_q$ is a non-orientable surface of genus q.[4]

Throughout this paper we may assume that no vertex of the graphs discussed has fewer than three arcs incident with it, and that such graphs contain no arcs whose end points coincide, or no pair of arcs forming a circuit. For, suppose v is a vertex on which there are only

* Presented in parts to the American Mathematical Society, under the titles "Irreducible Non-toral Graphs," April 19, 1935; and "Irreducible Non Projective Planar Graphs," (by title) Sept. 10 1935.

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[1] "Sur les Courbes Gauche," Fundamenta Mathematicae, Vol. XV, (1930), pp. 271–282. This result was first proved by P. A. Smith and Orrin Frink, (see Abstract 179, Bull. Amer. Math. Soc., Vol. 36, No. 3). The investigations of the pesent paper were undertaken at the suggestion of Professor Smith.

[2] H. Whitney, "Non Separable and Planar Graphs," Trans. Amer. Math. Soc., Vol. 34, pp. 339–362.

"Non Planar Graphs," Fund. Math. Vol. XXI (1933), pp. 73–84

[3] A graph is called irreducible with respect to a property of the graph if, when any arc of the graph is deleted, the graph loses that property.

[4] By the genus of a surface we mean the integer $p = (2 - K)/2$ when the surface is orientable, or the integer $q = 2 - K$ when the surface is non-orientable, where K is the characteristic.

two arcs *av*, *vb*. Then we may replace the chain *avb* by an arc *ab* without altering the mapability or non-mapability of the graph in question. If two arcs of a graph form a circuit we may delete one of these for the same reason. Similarly we may delete all arcs with coincident end-points.

The results of this paper are closely related to the general map coloring problem. Suppose we desire to know whether a map of q 2-cells, r or which are mutually touching, and with a given incidence relationship among them, exists on a surface S. The dual of this map will be the map of a graph G and will exist on the surface if and only if the map exists.[5] If such a map exists then we know that at least r colors are necessary for the coloring of this map.[6] The problem of determining if a given number of colors is necessary for the map on the surface in question is thus the same as determining whether a certain graph G can be mapped on this surface. Now having constructed all the irre-

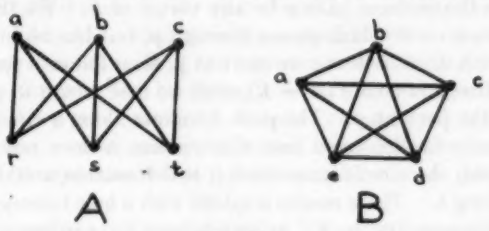

FIG. 1

ducible graphs which cannot be mapped on the surface, we can determine if G is among them, or contains one of them as subgraph.

Part I. The Maps of A and B on the Projective Plane and the Torus

The theorems of this part, while of interest in themselves, are needed in the proofs of the examples given in the latter parts of this paper.

§ 1. Maps of the graphs on the projective plane II.

Definitions. By a circuit of a graph G we mean a subgraph of G which is a simple closed curve. By a chain of G we mean a sequence of arcs $\xi_1, \cdots, \xi_n$, of G, such that ξ_i and ξ_{i+1} have an endpoint in common

[5] See Heffter, Math. Annalen, Vol. 38 (1891), p. 447, for a definition of dual maps and a proof of this statement.

[6] For an exposition of the map coloring problem see Philip Franklin, "A Six Color Problem," Journal of Math. and Physics, Vol. 13 (1934), p. 363.

for ($i = 1, \cdots, n-1$). A 2-cell is called regular if its boundary is a simple closed curve. A 2-cell α is called n-gonal if its boundary (denoted by $F(\alpha)$) is a circuit containing n vertices. By the degree of a vertex we mean the number of arcs with which it is incident. G is said to separate a surface S, on which it is mapped, if $S - G$ is not connected.

In a previous paper[7] we defined a surface as being maximal for a graph G, if it is the surface of greatest possible characteristic on which G can be mapped. For the purposes of the present paper we shall find it convenient to rephrase this concept as follows: An orientable ~~(nonorientable)~~ surface is called *minimal* for G if it is the ~~orientable (nonorientable)~~ surface of least possible genus on which G can be mapped. A surface which is maximal for G under the former definition is easily seen to be minimal of G under the new definition.

line 10-11 Correction see V. 19, 1940 P. 158

Theorem 1. *A and B separate* Π.

Proof: Let R be A or B. Suppose R does not separate Π when mapped on the surface. Let p be any vertex of R. We trace a simple closed curve λ on Π which passes through p, but has no other point in common with R, as follows; we start at p, then follow a path in a fixed arbitrary direction near $F(\Pi - R)$ until we first return to p. We then terminate the path at p. The path λ follows along a circuit of G, and can obviously be deformed into this circuit, λ does not separate Π, since if it did, the circuit into which it is deformable would separate Π.

Cut Π along λ. There results a sphere with a hole bounded by $\lambda_1 + \lambda_2$ each λ_i corresponding to λ. λ_1 and λ_2 contain vertices p_1 and p_2 respectively, each corresponding to p. If we close this hole with a 2-cell α, we obtain a sphere on which is mapped the graph $R^* + \lambda_1 + \lambda_2$, where R can be formed from R^* by coalescing the vertices p_1 and p_2. If we shrink $\alpha + F(\alpha)$ to a point, there will result a sphere on which R is mapped. Since R is non planar this gives a contradiction.

Theorem 2. *A or B separate* Π *into 2-cells.*

Proof: Π is homeomorphic with a sphere S with a cross cap[8] affixed.

[7] "The Triangulation of Surfaces and The Heawood Color Formula," Journal of Math. and Physics, Vol. XV (1936), pp. 179–186. We shall refer to this paper as T. We wish to call attention to an error in the paper T. The first sentence of the second paragraph of the introduction should read, "The Heawood color formula gives an upper bound $[P_k]$ for the number of colors sufficient to color a map on a surface of given characteristic, and hence for the number of mutually touching regions which can be mapped on this surface."

[8] A cross cap is a hemisphere with a circular hole, diametrically opposite points of the edge of the hole being identified. (B. v. Kerekjarto, "Vorlesungen über Topologie," p. 151 and fig. 28.)

Let R be A or B. Map R on S. If one of the regions into which S is separated is not a 2-cell, it is a 2-cell with a cross cap affixed. We can then remove the cross cap and cover the resulting hole with a 2-cell, thus mapping R on a sphere, which is impossible.

Lemma 1. *If A is mapped on* Π, *at least one of its circuits bounds.*

Proof: Let $\lambda = arcsbta$ be a circuit of A. If λ bounds, the lemma is true. If λ does not bound, cut the surface along λ. It will open out into a region homeomorphic with a circular disc α, with diametrically opposite points of its boundary identified. $A - \lambda$ consists of the arcs *sa*, *br*, *ct*. *sa* together with an arc of $F(\alpha)$ forms a circuit of A which bounds.

Lemma 2. *If B is mapped on* Π, *at least one of its circuits bounds.*

The proof is similar to that of lemma 1.

Lemma 3. *Let α be a 2-complex homeomorphic with a projective plane with a 2-cell deleted, and let a, b, c, d, be four points on $F(\alpha)$, such that the pair (a, b) separates the pair (c, d). Let ab and cd be non intersecting arcs in $\alpha - F(\alpha)$. Then $F(\alpha) + ab + cd$ separates α into two regular 2-cells.*

Proof: *ab* together with one of the two arcs into which the pair (a, b) separates $F(\alpha)$ forms a non bounding circuit, for otherwise *cd* would intersect *ab*. If we cut α along *ab*, it opens out into a 2-cell β, with boundary *acbadba*. *cd* separates β into two regular 2-cells.

Theorem 3. *A separates* Π *into one hexagonal and three quadrilateral 2-cells.*

Proof: The circuits of A contain either four or six arcs. Let α be a 2-cell, of those into which A separates Π, whose boundary is a circuit of A.

(1) If α is a quadrilateral, (we shall omit the word "2-cell" after the adjective "n-gonal" whenever no ambiguity results), say $F(\alpha) = arcta$. Temporarily delete *bs* from A. Then the region complementary to $\alpha + F(\alpha)$ consists of two hexagons with boundaries *arbtcsa* and *rcsatbr* respectively. Whichever of these *bs* is mapped in, is separated into two quadrilaterals by *bs*.

(2) If α is a hexagon, say $F(\alpha) = arbscta$. Temporarily delete *cr*. The region complementary to $\alpha + F(\alpha)$ consists of a quadrilateral with boundary *atsba*, and a hexagon with boundary *brasctb*. *cr* separates the latter into two quadrilaterals.

Fig. 2 gives a map of A on Π. The free edges are identified as indicated.

Theorem 4. *B separates* Π *in one of two ways,*

(*a*) *Four triangular and two quadrilateral 2-cells, or*

(*b*) *Five triangles and one pentagon.*

Proof: The circuits of B contain either three, four, or five arcs. The 2-cells into which Π is separated are thus either triangles, quadrilaterals or pentagons. Not all of these are triangles, for if this were true, we would have, if a_i denote the number of vertices of degree i,

$$a_2 = \frac{\sum i a_i}{3} = \frac{4 \cdot 5}{3}$$

which is not an integer.[9] Hence at least one 2-cell α is not triangular.

(1) If α is a quadrilateral, say $F(\alpha) = abcda$. Then by Lemma 3, if $\bar{\alpha}$ denote $\alpha + F(\alpha)$, $F(\alpha) + bd + ca$ separates the region complementary to $\bar{\alpha}$ into two quadrilaterals β and γ, where $F(\beta) = bdacb$, $F(\gamma) = bacdb$. If e is in β, then ea, eb, ec, ed separate β into four triangles. Similarly if e is in γ.

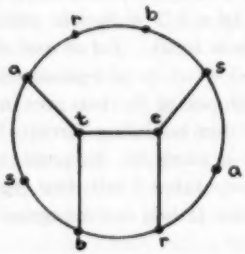

FIG. 2. Map of A on Π.

(2) If α is a pentagon, say $F(\alpha) = abcdea$. Let $\bar{\alpha} = \alpha + F(\alpha)$. By Lemma 3, $F(\alpha) + ac + bd$ separates the region complementary to $\bar{\alpha}$ into two 2-cells β and γ, where $F(\beta) = aedbca$, $F(\gamma) = dcabd$. Then eb and ec are in β and separate β into three triangles. ad is in γ and separates γ into two triangles.

Figure 3 represents the two types of maps. The free edges are identified as indicated.

Theorem 5. *If μ is any circuit of A, A can be mapped on Π so that one of the 2-cells into which Π is separated has μ for its boundary.*

Proof: A separates Π into three quadrilaterals and a hexagon. If μ is a quadrilateral circuit of A, either it bounds one of these 2-cells, or by suitably permuting the letters assigned to the vertices of A, μ will be the boundary of one of these quadrilaterals. Similarly if μ is a hexagonal circuit.

[9] T. Lemma 1.

Theorem 6. *If ν is any circuit of B, B can be mapped on* II *so that one of the* 2*-cells into which* II *is separated has ν for its boundary.*

Proof: By suitably permuting the letters assigned to the vertices in one of the two maps of fig. 3, ν will be the boundary of one of the 2-cells in one of these two maps.

§2. Maps of the graphs on the torus T.

Definitions. By an open path xy in a graph G, we mean a sequence of arcs $\xi_1, \cdots, \xi_n$, together with their endpoints, such that ξ_i and ξ_{i+1}, $(i = 1, 2, \cdots, n-1)$, have an endpoint in common, and $\xi_i \neq \xi_{i+1}$, $\xi_1 \neq \xi_n$, and such that x and y are the free endpoints of ξ_1 and ξ_n respectively. A closed path is a path whose initial point x coincides with its terminal point y. A path is called regular if it passes through

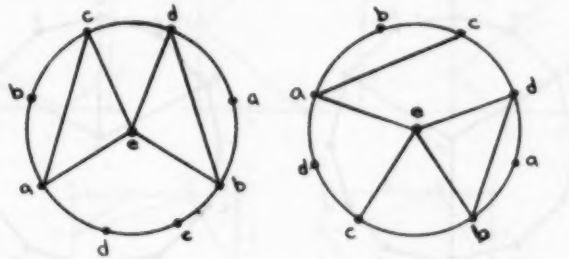

FIG. 3a FIG. 3b

FIG. 3. Maps of B on II.

each of its arcs and their endpoints exactly once, and is called irregular if it passes through some of its arcs or their endpoints more than once.

Theorem 7. *A and B separate* T *into* 2*-cells.*

Proof: T is minimal for each of these graphs. Hence the theorem follows.[10]

Lemma 4. *Let α be a* 2*-complex homeomorphic to a torus with a* 2*-cell deleted, and let a, b, c, d, be four points on $F(\alpha)$, such that the pair (a, b) separates the pair (c, d). Let ab and cd be non intersecting arcs in $\alpha - F(\alpha)$. Then $F(\alpha) + ab + cd$ bounds an irregular* 2*-cell.*

Proof: ab together with either one of the two arcs into which (a, b) separates $F(\alpha)$, forms a non bounding circuit, for otherwise cd would intersect ab. Cut α along ab. α will open out into a cylinder bounded by $j_1 = ab + acb$ and $j_2 = ab + adc$. If we cut this cylinder along cd,

[10] T. Theorem 1, corollary, and Theorem 2.

it will open out into a rectangular 2-cell with an irregular path for boundary.

Theorem 8. *A, when mapped on* T *separates it in one of two ways,*

(*a*) *Three regular hexagonal 2-cells, or*

(*b*) *Two regular quadrilaterals and an irregular hexagonal 2-cell with boundary of the form π = arbsctbrcsa.*

(Note: π is characterized by the fact that it is the sum of two regular chains *arbsct* and *tbrcsa*, each containing all the vertices of A.)

Proof: Let α be one of the 2-cells into which T is separated. $F(\alpha)$ must be a closed path of A. Since A contains no closed paths with an odd number of arcs, $F(\alpha)$ contains either four, six, or eight arcs of A.

(1) Suppose $F(\alpha)$ is a closed path of A containing four arcs, say $F(\alpha)$ = *arcta*. Delete for a moment the arc *bs*. Then the region complementary to α is an irregular 2-cell β with boundary *rcsarbtasctbr*. *bs* separates β into a regular quadrilateral and an irregular hexagon with boundary of the form π.

FIG. 4a

FIG. 4b

FIG. 4. Maps of A on T.

(2) Suppose $F(\alpha)$ is a closed path of A containing six arcs, say $F(\alpha)$ = *arbscta*. Delete for a moment the arc *bt*. Then the complementary region is an irregular 2-cell γ with boundary *rbsarctascr*. *bt* separates γ into two regular hexagons.

(3) Suppose $F(\alpha)$ contains eight arcs of A; say *at* is the arc of A not contained in $F(\alpha)$. Let $H = A - at$. Then $F(\alpha)$ must be a closed path passing through each arc of H either once, or twice in opposite directions, such that the path does not immediately retrace itself along any arc. There are essentially three such paths in H. (a) π_1 = *asctbrcsbtcra*, (b) π_2 = *arbtcrasctbsa*, (c) π_3 = *arbsctbrcsa*.

(a) If $F(\alpha)$ is π_1 then the complementary region is a regular quadrilateral with boundary *arbsa*. *at* cannot be in this complementary region, hence this case is impossible.

(b) Likewise it is impossible for $F(\alpha)$ to be π_2.

Hence (c). $F(\alpha)$ is π_3. The region complementary to $\alpha + F(\alpha)$ is then a 2-cell δ whose boundary is *ascbtcra*. *at* separates δ into two regular quadrilaterals.

Figure 4 represents the two types of maps. The free edges are identified as indicated.

Theorem 9. *B when mapped on* T *separates it in one of five ways;*

(*a*) *Four regular triangles and one irregular pentagon with boundary of the form abcadecda.*

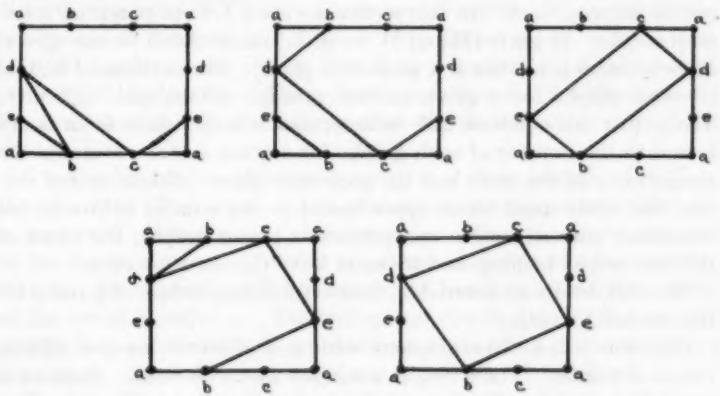

FIG. 5. Maps of B on T.

(*b*) *Three regular triangles, a regular quadrilateral, and an irregular pentagon with boundary of the form bcadecdb.*

(*c*) *Two regular triangles, one regular quadrilateral, and two regular pentagons.*

(*d*) *Two regular triangles, two regular quadrilaterals, and an irregular pentagon with boundary of the form aedceba.*

(*e*) *Five regular quadrilaterals.*

The proof is analogous to that of theorem 8.

Figure 5 represents the five types of maps. The free edges are identified as indicated.

Theorem 10. *If μ is any circuit of A, A can be mapped on* T *so that μ is the boundary of a regular* 2*-cell.*

Theorem 11. *If ν is any circuit of B, B can be mapped on* T *so that ν is the boundary of a regular 2-cell.*

By suitable permuting the letters assigned to the vertices in one of the maps of figures 4 or 5, μ or ν will be the boundary of a regular 2-cell.

The graphs A and B separate the projective plane into regular 2-cells. However they do not separate the torus, (as is seen from theorems 8 and 9), into regular 2-cells. In this connection it is of interest to note that the complete six-point separates the torus into regular 2-cells, and this latter graph is irreducible with respect to this property.[11]

Part II. Non Σ_p-al and non $\mathfrak{S}_q$-al graphs

If S is any surface, there exists a set of irreducible graphs which cannot be mapped on S. In this section we show how to construct a few such graphs. In parts III and IV we go into more detail for the special cases where S is a torus or a projective plane. The problem of finding all such graphs for a given surface presents considerable difficulty. The author has not been able to determine whether there is an upper bound to the number of such graphs for a given surface, even for the simple cases of the torus and the projective plane. While we feel certain that there must be an upper bound to the number of irreducible non toral, and irreducible non projective planar graphs, the proof of this has proved baffling, and we must leave this question open.

We shall denote an orientable, (non-orientable), surface of genus p by the symbol Σ_p, ($\mathfrak{S}_p$).

Definition. Let S be a sphere with a number of handles affixed. Delete a handle. There results a surface with two holes. Span each hole with a 2-cell. There results a sphere S' with one fewer handles than S. This process is called picking off a handle.

Theorem 1. *Let $G_1, G_2, \cdots, G_{n-2}, G_{n-1}, G_n$ be connected, irreducible non* Σ_p-al, Σ_q-al, $\cdots$, Σ_t-al, Σ_s-al, Σ_t-al *graphs respectively. Then*

$$G = G_1 + G_2 + \cdots + G_n$$

(where G_i does not intersect G_j, $i \neq j$) is an irreducible non Σ_k-al graph, where $k = (p + q + \cdots + r + s + t) + (n - 1)$.

Proof: G is non Σ_k-al. For, suppose G were Σ_k-al. Then G could be mapped on a sphere S with k handles. We show that this leads to a contradiction.

$G_2 + G_3 + \cdots + G_n$ separates S into a number of regions, and G_1 is mapped in α, one of these. Now since G_1 is non Σ_p-al, α must be a

[11] We omit the proof (which is empirical in character).

planar region with at least $(p + 1)$ handes affixed. For, if α were a planar region with fewer than $(p + 1)$ handles G_1 could not be mapped on it. Remove G_1, and then pick off $p + 1$ of the handles affixed to α. There results a sphere S' with $k - (p + 1)$ handles on which $G_2 + G_3 + \cdots + G_n$ is mapped.

Now $G_3 + G_4 + \cdots + G_n$ separates S' into a number of regions, and G_2 is mapped in α', one of these. Since G_2 is non Σ_q-al, α' must be a planar region with at least $(q + 1)$ handles affixed. Proceeding as in the preceding paragraph, we obtain a sphere S'' with $k - (p + 1) - (q + 1)$ handles, on which $G_3 + G_4 + \cdots + G_n$ is mapped.

We proceed in this manner, step by step until there results a sphere $S^{(n-2)}$ with $s + t + 1$ handles affixed[12] on which $G_{n-1} + G_n$ is mapped. Now G_n separates $S^{(n-2)}$ into a number of regions, in one of which, $\alpha^{(n-2)}$, G_{n-1} is mapped. $\alpha^{(n-2)}$ must be a planar region with at least $(s + 1)$ handles, since G_{n-1} is non Σ_s-al. Remove G_{n-1} and pick off $(s + 1)$ of the handles. There results a sphere $S^{(n-1)}$ with at most t handles, on which G_n is mapped. But this is impossible since G_n is non Σ_t-al.

G is also irreducible, for suppose an arc ξ of G_1 is deleted. Let $G_1' = G - \xi$. Map G_1' on an Σ_p. Let α be one of the regions into which Σ_p is separated. Affix $(q + 1)$ handles to α. Then G_2 can be mapped on the resulting region. Let β be one of the subdivisions of the resulting surface. By affixing a sufficient number of handles to β we can map G_3 on the resulting region. . . . Proceeding thus step by step we find that by adding $(k - p)$ handles to Σ_p we shall have obtained an Σ_k on which $G - \xi$ is mapped. Similarly if any arc of any of the other G_i is deleted.

Theorem 2. *Le P be an irreducible non Σ_p-al graph, and Q be an irreducible non Σ_q-al graph. Let G be the graph formed by coalescing any point x of P with any point y of Q. Then G is an irreducible non Σ_{p+q+1}-al graph.*

Proof: G is non Σ_{p+q+1}-al, for suppose G were Σ_{p+q+1}-al. Then G can be mapped on a sphere S with $(p + q + 1)$ handles. Now P separates S into a number of regions, and Q is mapped on one of these regions plus its boundary, (call it α), such that the point y of Q coincides

[12] The number of handles affixed to $S^{(n-2)}$ is

$$\begin{aligned} k - (p + 1) - (q + 1) - \cdots - (r + 1) &= (p + q + \cdots + r + s + t) \\ &\quad + (n - 1) - (p + 1) - \cdots - (r + 1) \\ &= s + t + n - 1 - (n - 2) \\ &= s + t + 1 \end{aligned}$$

with the point x of this boundary. Now the interior of α must be a planar region with at least $(q + 1)$ handles, for Q is non Σ_q-al. Remove Q, and then pick off $q + 1$ of the handles on α. There results a sphere with p handles on which P is mapped. But since P is non Σ_p-al, this is impossible.

G is also irreducible, for; suppose an arc ξ of P is deleted. Then we can map $P - \xi$ on an Σ_p. Let α be one of the regions, of those into which Σ_p is separated, having x on its boundary. Affix $q + 1$ handles to α. Then Q can be mapped on the resulting region in such a way that y is accessible to x. We can then coalesce x and y, and when this is done we shall have mapped $G - \xi$ on an Σ_{p+q+1}. Similarly if an arc of Q is deleted.

Definition. Let S be a sphere with a number of cross-caps. Delete a cross cap. There results a surface with a hole. Span this hole with a 2-cell. There results a sphere S' with one fewer cross-caps than S had. This process is called picking off a cross cap.

Theorem 3. *Let $G_1, G_2, \cdots, G_{n-2}, G_{n-1}, G_n$ be connected irreducible non $\mathfrak{S}_p$-al, $\mathfrak{S}_q$-al, $\cdots$, $\mathfrak{S}_r$-al, $\mathfrak{S}_s$-al, $\mathfrak{S}_t$-al graphs respectively. Then $G = G_1 + G_2 + \cdots + G_n$, (where G_i does not intersect G_j, $i \neq j$), is a non $\mathfrak{S}_k$-al graph, where $k = (p + q + \cdots + r + s + t) + (n - 1)$.*

(Note: In theorem 1 of this section, the analogous theorem for orientable surfaces, G is also irreducible. In this theorem G is not necessarily irreducible.)

Proof: G is non $\mathfrak{S}_k$-al, for suppose this were not true. Then G could be mapped on a sphere S with k cross-caps. We show that this leads to a contradiction.

$G_2 + G_3 + \cdots + G_{n-1} + G_n$ separates S into a number of regions, and G_1 is mapped in α, one of these. α must be a planar region with at least $p + 1$ cross-caps, for if α were a planar region with fewer than $p + 1$ cross-caps, G_1 could not be mapped on it. Remove G_1 and pick off $p + 1$ of the cross-caps affixed to α. There results a sphere S' with at most $k - (p + 1)$ cross-caps, on which $G_2 + G_3 + \cdots + G_n$ is mapped.

The balance of the proof is analogous to the proof of theorem 1 of this section the words "cross-cap" replacing the word "handle" in the discussion.

Theorem 4. *Let P be an irreducible non $\mathfrak{S}_p$-al graph, and let Q be an irreducible non $\mathfrak{S}_q$-al graph. Let G be the graph formed by coalescing any point x of P with any point y of Q. Then G is a non $\mathfrak{S}_{p+q+1}$-al graph.*

(The note after theorem 3 applies to this theorem also.)

The proof is analogous to that of theorem 2 of this section.

Part III. Irreducible Non Projective Planar Graphs

§1. Non accessible set of vertices of an $\mathfrak{S}_p$-al graph.

Definition. Let Q be an $\mathfrak{S}_p$-al graph. Suppose Q contains a set of vertices $(q_1, \cdots, q_r)$ such that no matter how Q is mapped on an $\mathfrak{S}_p$, not all of the vertices of this set are accessible to any one point of $(\mathfrak{S}_p - Q)$, and such that $(q_1, \cdots, q_r)$ is the minimal such set. We call such a set of vertices a non accessible set of order p for Q.

Theorem 1. *Let G be an irreducible non $\mathfrak{S}_k$-al graph. Delete an arc pq. Then the set of vertices (p, q) is a non accessible set of order k for $G - pq$.*

Proof: $G - pq$ is $\mathfrak{S}_k$-al, hence can be mapped on an $\mathfrak{S}_k$. If this could be done in such a manner that both p and q were accessible to the same point of $\mathfrak{S}_k - (G - pq)$ then p could be joined to q and G would then be mapped on an $\mathfrak{S}_k$.

Theorem 2. *Let G be an irreducible non $\mathfrak{S}_k$-al graph, and x any vertex of G. Let the arcs on x have for their other endpoints the vertices $(t_1, \cdots, t_n)$. Delete x and the arcs on x. Then $(t_1, \cdots, t_n)$ is a non accessible set of order k for the resulting graph G^*.*

Proof: G^* is $\mathfrak{S}_k$-al. If G^* could be mapped on an $\mathfrak{S}_k$ so that each of the vertices $(t_1, \cdots, t_n)$ is accessible to the same point y of $\mathfrak{S}_k - G^*$ then we could join y to each of these vertices and have a map of a graph homeomorphic with G on $\mathfrak{S}_k$.

In the sequel we shall be particularly interested in the following graphs; $H = A - ar$, $K = B - ab$, $L = B - (a; ab, ac, ad, ae)$, $M = A - (a; ar, as, at)$, where A and B are the two irreducible non planar graphs.

§2. Construction of irreducible non projective planar graphs. Let R_1, R_2, and R be irreducible non planar graphs. Let G be an irreducible non projective planar graph. Then

$$G = R + [a] + Q + [f],$$

where $[a]$ is a set of arcs, each of which has both its endpoints on R,

Q is a 1-complex not intersecting R,

$[f]$ is a set of arcs, called cut-arcs, each of which has one endpoint on R, and the other on Q. The endpoints of the arcs of $[f]$ which are on R are called terminals.

G contains at least one irreducible non projective planar graph as subgraph. Choose one of these and call it R. Once this choice of R is made we show that the structure of G, relative to R, is of one of the following types.

Type I. G is one of the three graphs $R_1 + R_2$, where R_1 does not intersect R_2.

Type II. Q is any one of the graphs H, K, L, M of $\S 1$. $[f]$ consists of arcs, one and only one of which is on each of the vertices of a non accessible set of vertices of Q, and such that the terminals are all contained in the same circuit of R.

Type III. The graphs of this type are characterized by the property; the terminals are all contained in some one circuit λ of R, $[a] = 0$, and Q is planar and does not contain as subgraph any of the graphs H, K, L, M.

Type III, (1). Let N be the graph composed of the arcs uw, ux, uy, uz, vx, vy, vz. Let G be formed by allowing the three vertices x, y, z of N to coincide with three distinct points contained in some circuit λ of R.

Type III, (2). Let w, x, y, z, be four distinct points of some circuit of R, where the pair (w, y) separates the pair (x, z) on this circuit, and on any other circuit of R containing these four points. Let Q be an arc uv. Let $[f] = (uw, uy, vx, vz)$.

Other subtypes. A few examples are given in Part V.

Type IV. The graphs of this type are characterized by the property; the terminals are not all contained in any one circuit of R, $[a] = 0$, and Q is planar.

There are three subtypes.

Type IV a. There is one terminal on each of three arcs of R, these three arcs all having an endpoint in common.

Type IV b. R is A, and there is one terminal on each of five arcs of $\boldsymbol{A}$, four of these arcs forming a circuit (say *arbsa*) and the fifth arc (say *ct*) having no endpoint on this circuit.

Type IV c. R is B, and there is one terminal on each of four arcs of B, three of these arcs forming a circuit, (say *abca*) and the fourth arc (say *de*) having no endpoints on this circuit.

Type V. The graphs of this type are characterized by the property that Q is planar and $[a] \neq 0$.

Theorem 3. *Let G be an irreducible non projective planar graph. G contains as subgraph an irreducible non planar graph R. Relative to R, the structure of G is of one of the types I–V.*

Proof: $G = R + [a] + Q + [f]$.

I). If Q is planar. (a) Suppose $[a] = 0$. There are two possibilities,

(1) The terminals are all contained in some circuit λ of R. If Q contains one of the graphs H, K, L, M as subgraph, then Q contains a

non accessible set of vertices and it is sufficient that Q be one of the graphs H, K, L, M, and that [f] consist of arcs, one of which is on each of the vertices of the non accessible set of Q. For, no matter how R is mapped on Π with λ the boundary of a 2-cell, Q cannot be mapped on this 2-cell so that all the vertices of the non accessible set are accessible to λ. G is then of type II.

If Q does not contain any of the graphs H, K, L, M as subgraph, then G is of type III.

(2) The terminals are not all contained in any one circuit of R. G is then of type IV.

(b) Suppose $[a] \neq 0$. Then G is of type V.

II) If Q is non planar, then for G to be non projective planar, it is sufficient that Q be an irreducible non planar graph. For, if Q is such it cannot be mapped on any of the 2-cells of $\Pi - R$. G is then of type I.

In Part V we give examples of the various types of irreducible non projective planar graphs.

Part IV. Irreducible Non Toral Graphs

§1. Non accessible set of vertices of an Σ_p-al graph. Let Q be a Σ_p-al graph. We define a non accessible set of vertices of order p for Q in the same manner as in §1 of Part III. The proofs of the following two theorems are identical with the proofs of theorems 1 and 2 of Part III.

Theorem 1. *Let G be an irreducible non Σ_p-al graph. Delete an arc pq. Then the set (p, q) is a non accessible set of vertices of order p for $G - pq$.*

Theorem 2. *Let G be an irreducible non Σ_p-al graph, and x any vertex of G. Let the arcs on x have for their other endpoints the vertices $(t_1, \cdots, t_n)$. Delete x and the arcs on x. Then $(t_1, \cdots, t_n)$ is a non accessible set of order p for the resulting graph G^*.*

We define the graphs H, K, L, M as in Part III.

§2. Construction of irreducible non toral graphs. We have

$$G = R + [a] + Q + [f],$$

where R, $[a]$, Q, $[f]$, are defined as before. As in Part III, we choose a subgraph R of G, and once this choice is made we show that the structure of G, relative to R, is of one of the following types.

Type I. G is one of the three graphs $R_1 + R_2$. (Where R_1 does not intersect R_2).

Type II. Let Q be any one of the graphs H, K, L, M. Let [f] con-

sist of arcs, one and only one of which is on each of the vertices of a non accessible set of vertices of Q, and let $[a] = 0$. The terminals are any points of R, not necessarily distinct. This type consists of all graphs having such a structure.

Type III. The graphs of this type are characterized by the properties; $R = A$, Q is planar and does not contain H, K, L, or M as subgraph, $[a] = 0$, and there are terminals on at most eight arcs of A.

Type IV. The graphs of this type are characterized by the properties; $R = B$, Q is planar and does not contain H, K, L, or M as subgraph, $[a] = 0$, and there are terminals on at most six arcs of B.

Type V. The graphs of this type are characterized by the properties; $R = A$, $[a] = 0$, Q is planar and does not contain any of the graphs H, K, L, or M as subgraph, and every arc of A contains at least one terminal.

Type VI. The graphs of this type are characterized by the properties; $R = B$, Q is planar and does not contain any of the graphs H, K, L, or M as subgraph, $[a] = 0$, and there are terminals on seven or more arcs of B.

Type VI, 1. There are terminals on each of seven arcs of B, such that the three arcs of B which do not contain terminals do not form a circuit or a chain.

Type VI, 2. There are terminals on eight or more arcs of B.

Type VII. The graphs of this type are characterized by the property that $[a] \neq 0$.

Theorem 3. *Let G be an irreducible non toral graph. G contains as subgraph an irreducible non planar graph R. Relative to R the structure of G is of one of the types I–VII.*

Proof: $G = R + [a] + Q + [t]$.

I. If Q is non planar. The proof is the same as for the analogous case in theorem 3 of Part III, G is of type I.

II. If Q is planar. (a) Suppose $[a] = 0$.

(1) If Q does not contain any one of the graphs H, K, L, or M there are four possibilities,

(i) R is A, and not all the arcs of A contain terminals. G is then of type III.

(ii) R is A and each arc of A contains at least one terminal. G is then of type V.

(iii) R is B and at most six arcs of B contain terminals. G is then of type IV.

(iv) R is B and seven or more arcs of B contain terminals. G is then of type VI.

(2) If Q contains a subgraph one of the graphs H, K, L, or M, then as in theorem 3 of Part III, we can show that it is sufficient that Q be one of these graphs. G is then of type II.

(b) If $[a] \neq 0$, G is of type VII.

In Part V we give examples of the various types of irreducible non toral graphs.

Part V. Examples of Irreducible Non Projective Planar and Irreducible Non Toral Graphs

§ 1. Examples of irreducible non projective planar graphs.

Type I. The examples are given in the description of this type.

Theorem: Any graph of type I is an irreducible non projective planar graph.

Proof: Since R_1 is non planar, it separates II into 2-cells when mapped on it. Since R_2 is non planar it cannot be mapped in any one of these 2-cells. Hence G is non p.p. (We shall use the abbreviation p.p. for the words "projective planar.") The irreducibility is obvious.

Definition: We shall employ the following symbolic representation for G,

$$G = (R/[a]/Q/[t]).$$

We shall indicate the fact that a terminal is on an arc xy of R by assigning to it the symbol p_{xy}, the subscript indicating which arc the terminal is on. To completely describe a graph we shall write out in full the arcs and vertices in each division of the symbol. If one of the components is absent in the graph, we shall place a 0 in the corresponding division of the symbol.

Type II. The only irreducible graphs formed as described are the following. All the other graphs formed as described are reducible.

$$G_{ii-1} = (A/0/uv, uw, ux, vw, vx, wx/us, vs, ws, xs)$$

$$G_{ii-2} = (A/0/uv, uw, ux, vw, vx, wx/ur, vr, wr, xs)$$

$$G_{ii-3} = (A/0/uv, uw, ux, vw, vx, wx/ur, vr, wt, xt)$$

$$G_{ii-4} = (A/0/uv, uw, ux, vw, vx, wx/ue, ve, wt, xt)$$

$$G_{ii-5} = (B/0/uv, uw, ux, vw, vx, wx/ue, ve, we, xe)$$

$$G_{ii-6} = (A/0/wx, vx, ux, yu, yv, yw/us, vs, ws)$$

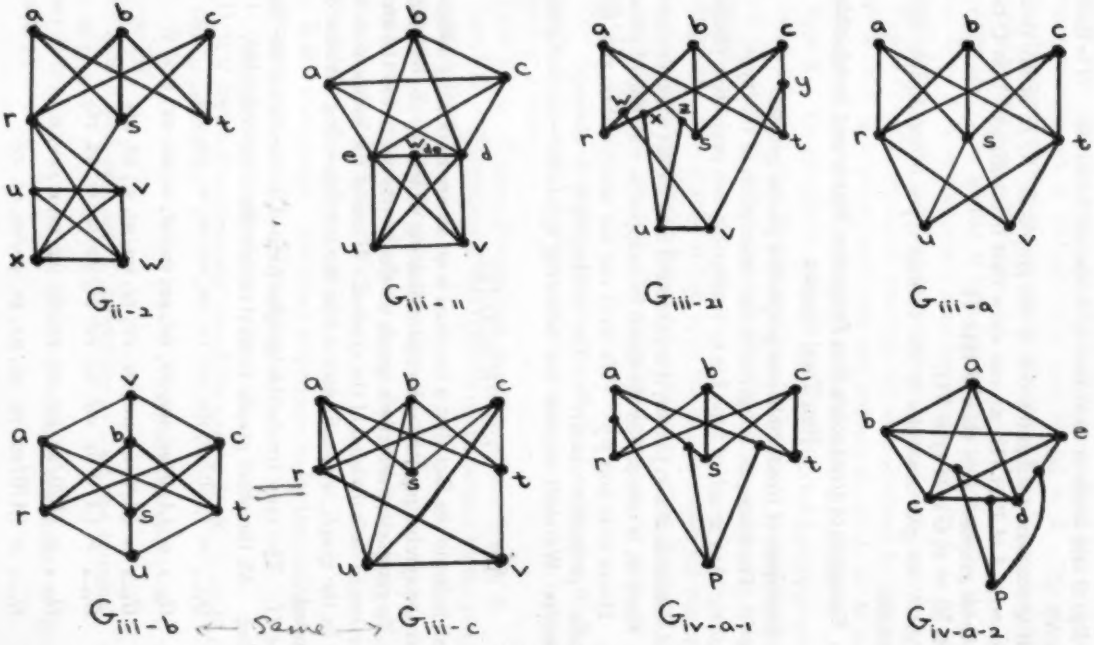
a b c r s t u v x w
G_{ii-2}
a b c e d u v
G_{iii-11}
a b c r s t u v y
G_{iii-21}
a b c r s t u v
G_{iii-a}
v a b c r s t u
G_{iii-b} ⟵ Same ⟶ G_{iii-c}
a b c r s t u v
a b c r s t p
G_{iv-a-1}
a b c d e p
G_{iv-a-2}

PLATE I

EXAMPLES OF IRREDUCIBLE NON PROJECTIVE PLANAR GRAPHS

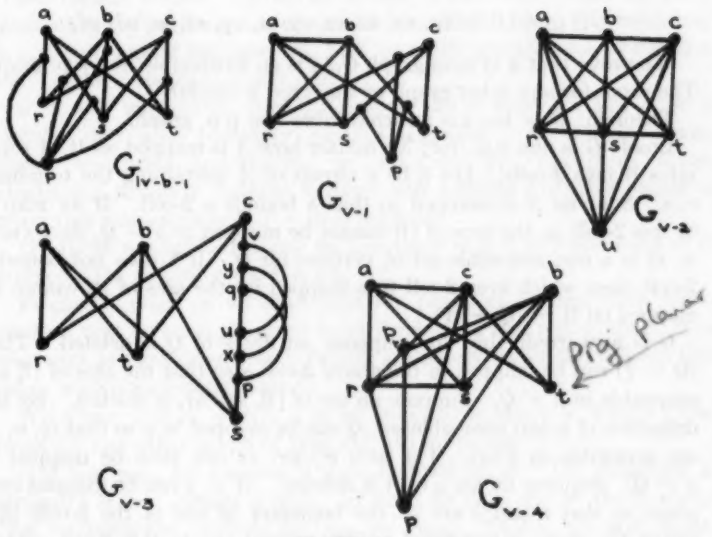

PLATE II

EXAMPLES OF IRREDUCIBLE NON PROJECTIVE PLANAR GRAPHS

$$G_{ii-7} = (A/0/xu, xv, xw, yu, yv, yw/ur, vr. ws)$$
$$G_{ii-8} = (B/0/xu, xv, xw, yu, yv, yw/ud, vd, wd)$$
$$G_{ii-9} = (A/0/xv, xw, yu, yv, yw, zu, zv, zw/xs, us)$$
$$G_{ii-10} = (A/0/xv, xw, yu, yv, yw, zu, zv, zw/zr, ul)$$
$$G_{ii-11} = (B/0/xv, xw, yu, yv, yw, zu, zv, zw/ud, xd)$$
$$G_{ii-12} = (A/0/ux, uv, uz, vx, vy, vz, xy, xz, yz/us, ys)$$
$$G_{ii-13} = (A/0/ux, uv, uz, vx, vy, vz, xy, xz, yz/ur, yt)$$
$$G_{ii-14} = (B/0/ux, uv, uz, vx, vy, vz, xy, xz, yz/ud, yd)$$

We prove that a typical graph G_{ii-2} is an irreducible non p.p. graph. The proof for any other graph of this type is similar.

Theorem: $G = G_{ii-2}$ is an irreducible non p.p. graph.

Proof: G is non p.p., for; No matter how A is mapped on Π, A separates Π into 2-cells. Let λ be a circuit of A containing the terminals r, s. Suppose A is mapped so that λ bounds a 2-cell. If we map Q in this 2-cell, α, the arcs of $[f]$ cannot be mapped in $\alpha - Q$, since (u, v, w, x) is a non accessible set of vertices for Q. If λ does not bound a 2-cell, then which ever 2-cell Q is mapped in, the arcs of $[f]$ cannot be mapped on $\Pi - (Q + A)$.

G is also irreducible, for; suppose an arc ξ of Q is deleted. Then $(Q - \xi)$ can be mapped in the above 2-cell α so that the arcs of $[f]$ are mappable in $\alpha - Q$. Suppose an arc of $[f]$, say ur, is deleted. By the definition of a non accessible set, Q can be mapped in α so that (v, w, x) are accessible in $F(\alpha)$. The arcs vr, wr, xs can then be mapped in $\alpha - Q$. Suppose an arc η of A is deleted. $A - \eta$ can be mapped on a plane so that r and s are on the boundary of one of the 2-cells into which the plane is separated. Affix a cross cap to this 2-cell. Then $Q + [f]$ can be mapped on the resulting region.

Type III.

Type III, (1). A few examples are:

$$G_{iii-11} = (B/0/uv/uw_{ds}, ud, ue, vw_{ds}, vd, ve)$$
$$G_{iii-12} = (B/0/uv/uw_{ds}, ub, uc, vw_{ds}, vb, vc)$$
$$G_{iii-13} = (B/0/uv/uc, ud, ue, vc, vd, ve)$$
$$G_{iii-14} = (A/0/uv/ua, uc, us, va, vc, vs)$$

Type III, (2). Two examples are

$$G_{iii-21} = (A/0/uv/ux_{rc}, uz_{as}, vy_{et}, vw_{rb})$$
$$G_{iii-22} = (B/0/uv/ux_{ad}, uz_{cd}, vw_{ac}. vy_{be})$$

Other examples of type III:

$$G_{iii-a} = (A/0/u, v/ur, us, ul, vr, vs, vl)$$
$$G_{iii-b} = (A/0/u, v/ur, us, ut, va, vb, vc)$$
$$G_{iii-c} = (A/0/uv/ua, ub, uc, vr, vt)$$

We prove that $G = G_{iii-11}$ is an irreducible non p.p. graph. The proof for any other graph of the type III, (1) is similar.

G is non p.p., for suppose G were p.p. The subgraph B of G separates Π into regular 2-cells. The arc uv must lie in one of these, and this 2-cell must have the vertices w_{de}, d, e on its boundary. Since this boundary is a circuit, one of the arcs uw_{de}, ud, ue must intersect one of the arcs vw_{de}, vd, ve. Hence G is non p.p.

G is also irreducible, for: (1) suppose uv is deleted. Let α be a circular disc with boundary $a_1c_1e_1v_1d_1a_2c_2e_2v_2d_2a_1$. In α map chains e_1uvd_1, and a_1bd_1, and arcs bc_1, be_1, ud_1, ue_1, vv_1, a_1e_2, d_1c_2. Upon identifying similarly lettered vertices and arcs on $F(\alpha)$, we obtain a projective plane on which $G - uv$ is mapped.

(2) In an analogous manner we can show that if any other arc of G is deleted the resulting graph is p.p.

From this point on we shall omit the proofs of irreducibility, when proving that an example given is an irreducible non p.p. graph. It is necessary to omit these details (which are purely empirical) to keep this paper within reasonable bounds.

We prove that $G = G_{iii-21}$ is an irreducible non p.p. graph. The proof for any other graph of type III, (2) is similar.

G is non p.p., for suppose G were mapped on Π. The subgraph A separates Π into regular 2-cells. uv must lie in α, one of these, where $F(\alpha)$ contains the vertices x_{rc}, z_{as}, y_{et}, w_{rb}. That is, the arcs rc, as, et, rb of A must all be contained in $F(\alpha)$, which is possible only if α is hexagonal, with boundary $F(\alpha) = tcrbsat$. The pair of vertices (x_{rc}, z_{as}) then separates the pair (y_{et}, w_{rb}) on this boundary. Hence chain $x_{rc}uz_{as}$ intersects chain $y_{et}vw_{rb}$. That is, G is non p.p.

The irreducibility of G is shown in the general manner of deleting an arc at a time, and mapping the resulting graph on a circular disc plus its boundary, diametrically opposite points of the boundary of the disc being identified (as in the proof for G_{iii-11}).

The proof that G_{iii-a} is an irreducible non p.p. graph is similar to the proof for graphs of type III, (1). The proof for G_{iii-b} is also similar to that of the above type. The proof for G_{iii-c} is similar to the proof for G_{iii-21}.

Type IV.

Type IV, a. There are two such graphs.

$$G_{iv-a-1} = (A/0/p/pp_{at}\ ,\ pp_{as}\ ,\ pp_{ar})$$

$$G_{iv-a-2} = (A/0/p/pp_{bd}\ ,\ pp_{dc}\ ,\ pp_{da})$$

Type IV, b. A typical graph is

$$G_{iv-b-1} = (A/0/p/pp_{as}\ ,\ pp_{bs}\ ,\ pp_{br}\ ,\ pp_{ra}\ ,\ pp_{ct})$$

Type IV, c. No irreducible graph of this type exists.[13]

We now prove that $G = G_{iv-a-1}$ is an irreducible non p.p. graph. The proof for G_{iv-a-2} is similar.

G is non p.p., for suppose G could be mapped on II. The subgraph A separates II into regular 2-cells, hence the vertex a appears on the boundary of one of these either once or not at all. Hence at most two of the arcs ar, as, at are on the boundary of a given one of these 2-cells. Hence no matter in which of these 2-cells p is mapped in, at least one of the arcs on p intersects A.

The irreducibility is shown in the usual way.

We show that $G = G_{iv-b-1}$ is an irreducible non p.p. graph.

G is non p.p., for suppose G could be mapped on II. The arcs of A on which there are terminals are not all contained in any one circuit of A. Hence no matter how the subgraph A happens to be mapped on II, not all the terminals are on the boundary of any one 2-cell. Hence no matter which one of the 2-cells into which A separates II p happens to be in, at least one arc on p intersects A.

The irreducibility is shown in the usual way.

Type V. A few examples are,

$$G_{v-1} = (A/ab,\ rs/p/pp_{cr}\ ,\ pp_{cs}\ ,\ pp_{st})$$

$$G_{v-2} = (A/ab,\ bc,\ rs,\ st/u/ua,\ uc,\ ur,\ us,\ ut)$$

$G_{v-3} = (A'/pq,\ uv,\ xy/0/0)$, where the vertices p, q, u, v, x, y are on arc cs of A, and the order of the vertices on cs is $spxuqyvc$.

$$G_{v-4} = (A/ac,\ rs,\ bp_{rc}/p/pb,\ pc,\ pr,\ pp_{rc}).$$

[13] The author has been able to show empirically that any graph of this type is reducible. For example, a graph of this type is

$$G = (B/0/p/pp_{ab}\ ,\ pp_{bc}\ ,\ pp_{ca}\ ,\ pp_{da}).$$

G contains as subgraph a graph homeomorphic with the irreducible non p.p. graph

$$G' = (A/ap_{ar}\ ,\ bp_{ar}\ ,\ cp_{br}\ ,\ rs/0/0)$$

which is a graph of type V.

$G = G_{v-1}$ is an irreducible non p.p. graph.

Proof: The only possible circuits of A containing the three arcs cr, cs, bt are the hexagonal circuits $\lambda_1 = scrbtas$ and $\lambda_2 = scratbs$. Map A on II. Since p is joined to p_{cr}, p_{cs}, p_{bt}, in G, if G were p.p. p would have to be in a 2-cell α, of those into which A separates II, such that $F(\alpha)$ contains the arcs cs, rc, bt, which is possible only if α is a hexagonal 2-cell, and $F(\alpha)$ is λ_1 or λ_2. In either case one of the quadrilateral 2-cells β is then bounded by $\lambda_3 = sarbs$. Now rs and ab cannot both be in β, hence one of these arcs must be in α. If ab is in α then chain $p_{cs}pp_{bt}$ intersects ab. If rs is in α then chain $p_{cs}pp_{bt}$ intersects rs. Hence G is non p.p.

The irreducibility of G is shown in the usual manner.

The proof for G_{v-6} is similar. G_{v-3} is equivalent to a graph of type III, (2). We thus see that there is duplication among the types, so that they do not constitute a mutually exclusive classification.

$G = G_{v-4}$ is an irreducible non p.p. graph.

Proof: G contains as subset the graph B^* formed by joining the vertices a, b, c, s, t to each other, (where the chain arb joins a to b, and the chain bpc joins b to c, to form B^*). Map B^* on II. II is separated into regular 2-cells. Since p_{rc} is joined to each of the vertices r, b, p, c in G, if G were p.p., p_{rc} would have to lie in one of these 2-cells α, such that $F(\alpha)$ contains the chain $arbpc$. Now rp cannot be in α without intersecting either chain $p_{rc}qb$ or arc $p_{rc}c$. Also rp cannot be in the complementary region without intersecting the arc bs or bt. Hence G is non p.p.

The irreducibility is shown in the usual manner.

§ 2. Examples of irreducible non toral graphs. We shall employ the same symbolic representation for an irreducible non toral graph as was used for an irreducible non projective planar graph in § 1. In proving that a non toral graph is irreducible, we shall employ the following method (except for graphs of type II). We shall represent the torus by a rectangular 2-cell α, the opposite edges of which are E_1, E_2 and F_1, F_2, such that $F(\alpha) = E_1 + F_1 - E_2 - F_2$. Upon identifying E_1 with E_2, and F_1 with F_2, the resulting surface obtained is a torus. Such a rectangular 2-cell we shall call a torocell. Suppose the vertices of E_1 are u, v, w, x, that is, suppose $E_1 = E_2 = uvwxu$ then by the notation v_1, w_1, etc., we shall mean the vertices v, w, etc. considered as vertices of E_1, while by v_2, w_2, etc., we shall mean these same vertices considered as being on E_2. Similarly by the notation u_{11}, u_{21}, etc., we shall mean the vertex u considered as the vertex at the

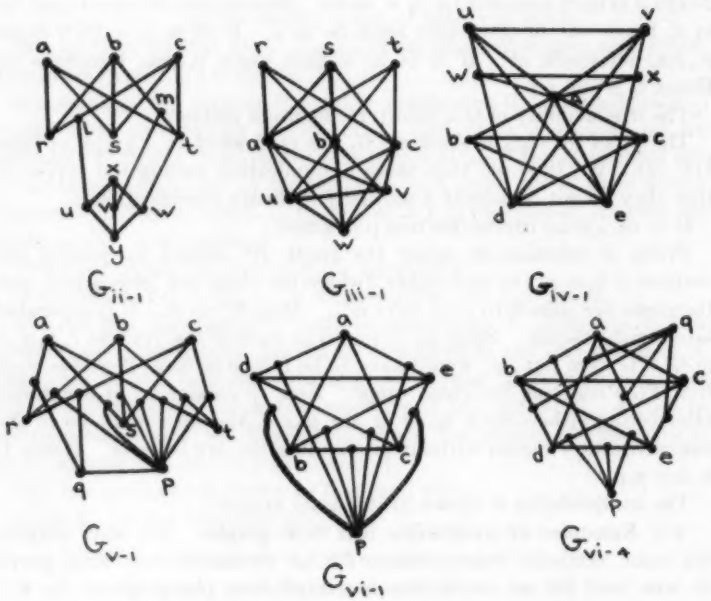

PLATE III

EXAMPLES OF IRREDUCIBLE NON TORAL GRAPHS

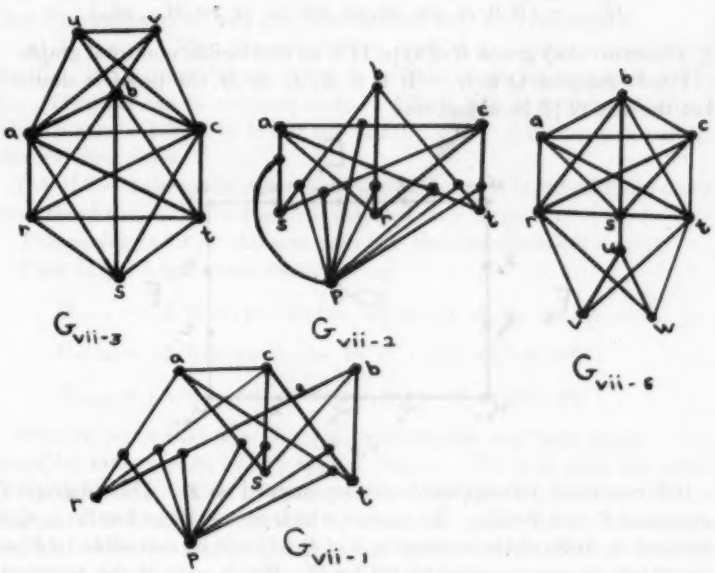

PLATE IV

EXAMPLES OF IRREDUCIBLE NON TORAL GRAPHS

intersection of E_1 with E_2, (or E_2 with F_1, etc.). (See Fig. 6.) Similar notation shall be used for the vertices on F_i.

Type I. The examples are given in the description of this type.

Type II. The examples given in § 1, of irreducible non projective planar graphs, are also irreducible non toral graphs. Two additional examples are,

$$G_{ii-1} = (A/0/xu,\ xv,\ xw,\ yu,\ yv,\ yw/ul_{rc}\ ,\ vm_{bl}\ ,\ wt)$$

$$G_{ii-2} = (B/0/xv,\ xw,\ yu,\ yv,\ yw,\ zu,\ zv,\ zw/xl_{ac}\ ,\ ul_{ac})$$

Theorem: Any graph G of type II is an irreducible non toral graph.

Proof: Suppose Q is H. (If Q is K, L, or M the proof is similar.) Let the arcs of $[t]$ be al and rm.

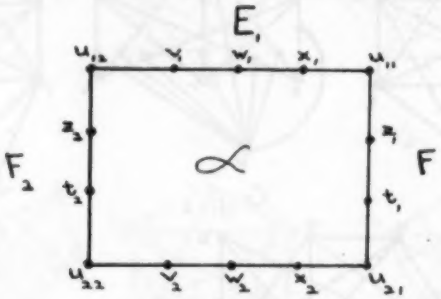

FIG. 6

G is non toral, for suppose G can be mapped on T. The subgraph R separates T into 2-cells. No matter which one of these 2-cells, α, Q is mapped in, both of the vertices a, r of Q will not be accessible to $F(\alpha)$, since they are a non accessible set for Q. Hence even if the terminals l and m are on $F(\alpha)$, both the arcs al and rm cannot be mapped in $\alpha - Q$, without intersecting Q; hence G is non toral.

G is also irreducible, for: (1) suppose an arc of $[t]$ is deleted, say al. Map R on a torus so that one of the 2-cells into which the surface is separated has the terminal m on its boundary. Map the circuit $\lambda = btesb$ of Q in this 2-cell α. λ separates α into a 2-cell β and a ring shaped region γ. Map chain tas in β and chain brc in γ. Then r can be joined to m. $G - al$ is thus toral. Similarly $G - rm$ is toral.

(2) Suppose an arc of Q is deleted, say bs. Then map R on a torus T so that one of the 2-cells into which T is separated has both the terminals l and m on its boundary. Map the circuit $\mu = cratc$ of Q in this 2-cell

α. μ separates α into a 2-cell β and a ring shaped region γ. Map chain *rbt* in β and chain *asc* in γ, in such a way that t is interior to the circuit *crasc*. Then a and r are accessible to l and m respectively Hence $G - bs$ is toral. Similarly if any other arc of Q is deleted.

(3) Suppose an arc of R^* is deleted. (Where R^* is the graph R with the terminals l and m considered as vertices of degree two in R.) Let ξ be the deleted arc. Then $R^* - \xi$ is planar. Map it on a sphere, which is then separated into 2-cells. Let δ be one of these 2-cells such that $F(\delta)$ contains m, and ϵ be a 2-cell such that $F(\epsilon)$ contains l.

(a) If $\delta \neq \epsilon$. Map the circuit $\lambda = btcsb$ in δ. λ separates δ into a 2-cell δ_1 and a ring shaped region δ_2. Map chain *tas* in δ_1 and *brc* in δ_2. Then *rm* can be mapped in δ_2. Affix a handle bridging δ_1 and ϵ Then a can be joined to l over this handle. $G - \xi$ is then mapped on the resulting torus.

(b) If $\delta = \epsilon$ then affix a handle bridging δ_1 and δ_2 in (a) and join a to l over this handle. $G - \xi$ is then mapped on a torus.

This accounts for all the arcs, and G is thus irreducible.

Type III. A few examples are

$$G_{iii-1} = (A/0/uv, vw, wu/ua, ub, uc, va, vb, vc, wa, wb, wc)$$

$$G_{iii-2} = (A/0/u, vw/ur, us, ut, vr, vs, vt, wa, wb, wc)$$

$$G_{iii-3} = (A/0/u, vw/ua, ub, uc, vr, vs, vt, wr, ws, wt)$$

We now prove that $G = G_{iii-1}$ is an irreducible non toral graph. The proof for other graphs of this type is similar. We shall omit the proof of irreducibility, which is analogous to that for G_{viii-1} below. (It is necessary to omit these details to keep this paper within reasonable bounds. For the graph G_{viii-1} we shall give an irreducibility proof to show the details of such a proof, and shall omit the proofs for all the other examples, as they follow the same general lines.)

$G = G_{iii-1}$ is non toral, for suppose G were toral. (a) Suppose the subgraph A separates the torus T into three regular hexagons. Since u, v, w are each joined to each other, and to each of the vertices a, b, c of A, they must all lie in the same one of these 2-cells α. Then chain *auvb* separates α into 2-cells α' and α'' where α' is the 2-cell with c on its boundary. Since w is joined to c in G, w must be in α'. Then chain *awv* intersects *uc*.

(b) If A separates T into two regular quadrilaterals and an irregular hexagon α, then since all three of the vertices a, b, c cannot be on the boundary of the same quadrilateral, u, v, w must lie in α. $F(\alpha)$ will

be of the form *atbscrbtcsa*. The vertex a of the triad (a, b, c) of A appears but once on $F(\alpha)$. Since a is joined to u, v, w, either (1) uw intersects vb or va, or (2) uv intersects wb or wa, or (3) vw intersects ub or ua.

Hence G is non toral.

Type IV. A typical example is

$$G_{iv-1} = (B/0/uv, vw, wx, xu/ua, ue, va, vd, xa, xe, wa, wd)$$

The proof that this an irreducible non toral graph is analogous to that for G_{vii-1} below.

Type V. A typical example is

$$G_v = (A/0/pq/pp_{rb}, pp_{bs}, pp_{cs}, pp_{as}, pp_{at}, pp_{bt}, pp_{ct}, qq_{ar}, qq_{rc})$$

This graph is non toral, for; no matter how A is mapped on $\mathbf{T}$, at most eight arcs of A are on the boundary of any one of the 2-cells into which $\mathbf{T}$ is separated. Since p is joined to q, pq must lie in one of these. Hence at least one of the arcs on p or on q must intersect A.

We omit the proof of irreducibility.

Type VI.

Type VI, 1. Three examples are,

$$G_{vi-1} = (B/0/p/pp_{ad}, pp_{as}, pp_{bc}, pp_{bd}, pp_{bs}, pp_{cd}, pp_{cs})$$

$$G_{vi-2} = (B/0/p/pp_{as}, pp_{ad}, pp_{ac}, pp_{bd}, pp_{cd}, pp_{cs}, pp_{ds})$$

$$G_{vi-3} = (B/0/pq/pp_{bc}, pp_{cd}, pp_{ds}, pp_{bs}, qq_{bd}, qq_{as}, qq_{cs})$$

Type VI, 2. A typical example is

$$G_{vi-4} = (B/0/p, q/pp_{ab}, pp_{bs}, pp_{ds}, pp_{cd}, qq_{bd}, qq_{ad}, qq_{as}, qq_{cs})$$

We prove that $G = G_{vi-4}$ is non toral. The proof for other examples of type VI follows the same general lines. We omit the proof of irreducibility.

Consider the chain $\xi = abdec$ of B. Map B on a torus $\mathbf{T}$. If ξ is not on the boundary of a 2-cell, of those into which $\mathbf{T}$ is separated, then p is not accessible to all the terminals $p_{ab}, p_{bs}, p_{cd}, p_{dc}$. Suppose ξ is on the boundary of a regular 2-cell α. Then $F(\alpha)$ must be *abedca*. Let $c(\alpha)$ be the portion of $\mathbf{T}$ complementary to $\alpha + F(\alpha)$. Then *ad* and *bc* satisfy the conditions of lemma 4, Part I, and $c(\alpha)$ is thus an irregular 2-cell. The arcs *ea*, *bd*, and *ce* separate $c(\alpha)$ into 2-cells such that no one of these has all four of the terminals $q_{bd}, q_{ad}, q_{as}, q_{cs}$ on its boundary. q is thus inaccessible to all of these terminals. If ξ is on

the boundary of an irregular 2-cell β, then no one of the 2-cells of $c(\alpha)$ has all the terminals q_{ij} on its boundary.

Hence G is non toral.

Type VII. A few examples are

$G_{vii-1} = (A/ac/p/pp_{ar}, pp_{as}, pp_{at}, pp_{br}, pp_{cr}, pp_{cs}, pp_{ct})$

$G_{vii-2} = (A/ac/p/pp_{ac}, pp_{as}, pp_{at}, pp_{br}, pp_{bs}, pp_{bt}, pp_{cs}, pp_{ct})$

$G_{vii-3} = (A/ab, ac, bc, rs, rt, st/uv/ua, ub, uc, va, vb, vc)$

$G_{vii-4} = (A/ab, bc, rs, rt/uv/ua, uc, ur, us, ut, va, vc, vr, vs, vt)$

$G_{vii-5} = (A/ab, bc, ac, rs, rt, st/u, v/ua, uc, ur, su, ut, va, vc, vr, vs, vt)$

$G_{vii-6} = (A/ab, bc, ac, rs, rt/uv, uw/us, vr, vs, vt, wr, ws, wt)$

$G_{vii-7} = (A/ab, bc/uv, vw, wu/ua, us, uc, va, vb, vs, wr, wb, wt)$

We prove that $G = G_{vii-1}$ is an irreducible non toral graph.

G is non toral, for suppose G can be mapped on a torus T.

(1) Suppose the subgraph A of G is mapped on T so that T is separated into three regular hexagons. Then no matter which of these p is mapped in it is accessible to at most six terminals.

(2) Suppose A separates T into two regular quadrilaterals and an irregular hexagon α. One of the arcs of A is not contained in $F(\alpha)$. If p is in one of the quadrilaterals it is accessible to at most four terminals. If p is in α; if the arc of A not contained in $F(\alpha)$ has a terminal on it then p cannot be joined to this terminal; hence suppose the arc of A not contained in $F(\alpha)$ is either bs or bt.

(a) Suppose bs is not in $F(\alpha)$. Since the boundaries of the quadrilaterals cannot have all three vertices a, b, c on any one of them, ac must be in α. $F(\alpha)$ is either *satcrbtarcs* or *sarctbratcs*.

If $F(\alpha)$ is the former, then ac intersects one of the chains $p_{br}pp_{ct}$, $p_{br}pp_{ar}$, or $p_{cs}pp_{cr}$. If $F(\alpha)$ is the latter a similar situation exists.

(b) Suppose bt is not in $F(\alpha)$. Since G is symmetric with respect to the arcs bs and bt, the discussion is the same as for (a).

Thus G cannot be mapped on T.

G is also irreducible, for; (1) Suppose the arc pp_{cr} is deleted. Let α be a torocell, where $E_1 = E_2 = csac$ and $F_1 = F_2 = crbtc$. In α map arcs b_2s_1, t_2a_1, a_1r_1. These separate α into four 2-cells, one of which, β, is bounded by *arbtcascta*. Map p in β. Then p is accessible to every terminal except p_{cr}. Hence $G - pp_{cr}$ is toral. If cp_{cr} is deleted in the above map then chain $pp_{cr}r_1$ can be mapped in β. Hence $G - cp_{cr}$ and $G - p_{cr}r$ are toral.

In an analogous manner we can show that if any other arc of G is deleted the resulting graph is toral.

The proof that G_{viii-2} is an irreducible non toral graph is similar to that for G_{viii-1}

We prove that G_{viii-3} is non toral.

Suppose $G = G_{viii-3}$ were toral. G contains as subgraph the complete six-point H with vertices a, b, c, r, s, t. No matter how H is mapped on T, T is separated into regular 2-cells. Suppose α is one of these such that $F(\alpha)$ contains a, b, and c. Then chain $buvc$ separates α into two 2-cells γ and β, where β is that 2-cell with a on its boundary. Then in β bv must intersect ua. Hence G is non toral. We omit the proof of irreducibilty

We prove that $G = G_{viii-4}$ is non toral.

Suppose G were toral. If the subgraph A separates T into three regular hexagons, then which ever one of these uv is in, u and v cannot both be joined to each of the vertices a, c, r, s, t, without some of the arcs on u and v intersecting. If T is separated into two regular quadrilaterals and an irregular hexagon α, then uv must be in α for both u and v to be accessible to all the terminals. At least two of the arcs ab, bc, rs, st must be in α. These two arcs separate α into three 2-cells, such that which ever one of these uv is in, both u and v cannot be joined to all the terminals. Hence G is non toral. We omit the proof of irreducibility

The proofs for G_{viii-5} and G_{viii-6} follow along the same general lines as for G_{viii-3}.

Finally, we prove that $G = G_{viii-7}$ is non toral.

G contains as subgraph the graph $\bar{A}$ formed by joining each vertex of the triad (u, r, t) to each vertex of the triad (w, c, a). Map $\bar{A}$ on a torus T. Either

(1) T is separated into three regular hexagons; since v, s, b are joined to each other on G, the must all lie in the same one of these 2-cells if G is toral. The arcs br, bt, bw, bc, ba separate this 2-cell into 2-cells no one of which has all the vertices u, w, c, a on its boundary. Hence no matter which one of these v and s are mapped in, they are not accessible to all the vertices to which they are joined in G. Hence G cannot be toral in this case.

or (2) T is separated into two regular quadrilaterals and an irregular hexagon α with boundary of the form

$$l\ p_1\ p_2\ p_3\ p_4\ m\ q_1\ q_2\ q_3\ q_4\ l,$$

where l is a vertex of the triad (u, r, t), m is a vertex of the triad (w, c, a) the p's and q's are all distinct, and the p's and q's are the vertices of $\bar{A}$ distinct from l and m. Since b is joined to w, c, a, r, t in G, b cannot be mapped in one of the quadrilaterals, and hence must be in α. Since b is joined to v and s in G, v and s are also mapped in α.

(a) If l is r. Then chain mbr separates α into two regular 2-cells. Since v is joined to s in G, v and s are both in the same one of these. Then either as intersects chain bvu or va intersects chain bsu. Similarly if l is t.

(b) If l is u, and m is w. Then chain wvu separates α into two regular 2-cells. s and b must be in the same one of these, since they are joined to each other and to v in G. Then either chain wbc intersects sa or chain vsc intersects ba. Similarly if l is u and m is c or a.

G is thus non toral. We omit the proof of irreducibility.

FOUR PLACE TABLE OF THE BESSEL FUNCTION

By M. M. Clapp

Values of $J_0(x)$ and $J_1(x)$ computed from the series expansion (omitting terms of order x^{-2}). Accuracy is $\pm.0002$. Underlined values are values for the roots, maxima, and minima of $J_0(x)$ and the corresponding values of $J_1(x)$ from Jahnke, E., and Emde, F., *Funktionentafeln, Tables of Functions:* second (revised) edition, p. 237, 1933. For values of x $(0<x<25.1)$ see Hayashi, K., *Tafeln der Besselschen, Theta-, Kugel- und anderer Funktionen:* pp. 2–27, 1930.

x	$J_0(x)$	$J_1(x)$	x	$J_0(x)$	$J_1(x)$	x	$J_0(x)$	$J_1(x)$
25.3	+.1293	−.0900	30.6346	0	−.1442	35.6	−.1294	−.0353
25.6	+.1496	−.0471	30.9	+.0376	−.1379	35.9	−.1133	−.0716
25.9037	+.1567	0	31.2	+.0765	−.1194	36.2	−.0872	−.1011
26.2	+.1499	+.0454	31.5	+.1083	−.0905	36.5	−.0535	−.1206
26.5	+.1299	+.0870	31.8	+.1300	−.0537	36.8	−.0154	−.1306
26.8	+.0988	+.1203	32.1897	+.1406	0	36.9171	0	−.1313
27.1	+.0588	+.1426	32.4	+.1376	+.0292	37.2	+.0365	−.1251
27.4935	0	+.1522	32.7	+.1227	+.0683	37.5	+.0717	−.1078
27.8	−.0457	+.1440	33.0	+.0973	+.1007	37.8	+.1003	−.0811
28.1	−.0857	+.1221	33.3	+.0634	+.1239	38.1	+.1197	−.0473
28.4	−.1179	+.0902	33.6	+.0241	+.1358	38.4748	+.1286	0
28.7	−.1391	+.0507	33.7758	0	+.1373	38.7	+.1254	+.0292
29.0468	−.1480	0	34.0	−.0304	+.1329	39.0	+.1114	+.0640
29.3	−.1433	−.0369	34.3	−.0682	+.1169	39.3	+.0875	+.0935
29.6	−.1261	−.0771	34.6	−.0995	+.0907	39.6	+.0561	+.1144
29.9	−.0978	−.1115	34.9	−.1219	+.0566	39.9	+.0199	+.1250
30.2	−.0611	−.1327	35.2	−.1330	+.0178	40.0584	0	+.1261
30.5	−.0194	−.1435	35.3323	−.1342	0			

REACTION VELOCITY IN THE SYSTEM: $SOLID_I + GAS \rightarrow SOLID_{II}$

II. Flow Systems

By George S. Gardner*

In a previous paper[1] the writer has studied the velocity in the reaction between a granular solid and a gas, for the special case in which the product of the reaction is a second solid phase, and the actual rate of reaction of the gas and solid *at* the reacting surface is great compared to the rate at which the reactant gas is *transferred to* the reacting surface. An equation has been developed for the velocity of this type of reaction in closed systems, which has been found to be quite generally applicable. It now remains to apply this equation to flow systems.

However, it is evident that conditions in flow systems are far more complex than those existing in closed systems. In closed systems, for a given value of the time, the partial pressure is practically uniform throughout the reaction vessel, and each particle of solid has reacted to approximately the same degree with the gas. At least, this is true if every particle of the solid is as easily accessible to the gas as every other particle. In flow systems, on the other hand, there are other complicating factors. If the state is steady, the partial pressure of the reactant gas varies from point to point in the tube, and the concentration of solid reaction product does likewise. If the state is not steady, we have, in addition to the above, a variation with time at every point in the tube. Although these additional difficulties are present, we can, nevertheless make some progress toward their solution, as the writer will attempt to show.

Consider a tube of circular cross section, filled with small, homogeneous, approximately spherical granules of solid absorbent. Through this tube is flowing a mixture of gases. One of these gases reacts with the solid, and the product of the reaction is a second solid phase. In many reactions of this type the reaction of the solid and the reacting gas is very rapid after the gas actually reaches the solid reactant sur-

* Contribution from the Laboratory of Brewer & Gardner, Philadelphia.

[1] Gardner, J. Math. Phys., Mass. Inst. Tech., 14, 241 (1935).

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face. The controlling factor is thus the diffusion of the reactant gas through the layer of solid reaction product existing on each granule of solid.

In such a procedure, we find that what occurs is characteristic of this type of process. Figure (1) represents conditions in a tube such as

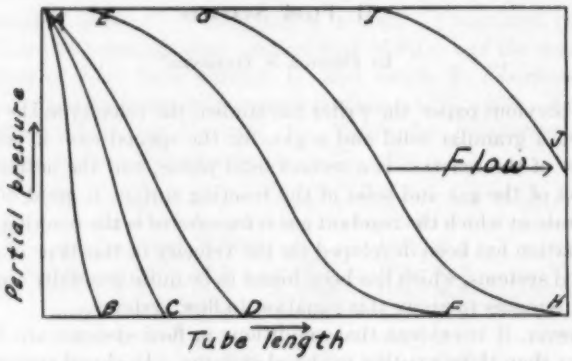

FIG. 1

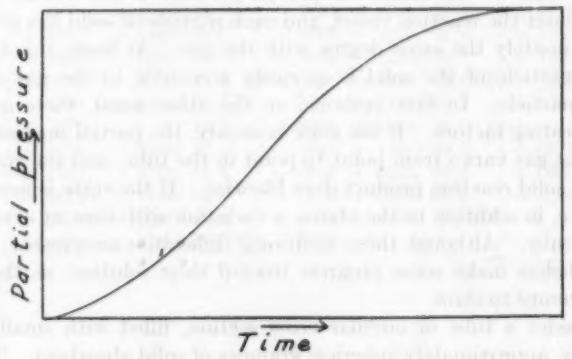

FIG. 2

described above. The abscissae represent linear distance in the tube, in the direction of gas flow, measured from the point of entrance of the gases into the mass of solid particles. The ordinates represent partial pressure of the reactant gas in the gas stream. At the earliest portion of the run we have a curve similar to AB. That is, the partial pressure

of the reactant gas drops off very sharply with distance. As time goes on, the curve changes to AC, and then to AD, and after the initial portion of the solid is practically saturated with the gaseous reactant, we get the curve EF. This corresponds to the establishment of equilibrium conditions for our process, and a curve of the form EF will progress in the direction indicated, without change in form (GH, IJ, etc.).

In the same way, if we study conditions at a given point in the tube, we find that the change of partial pressure with time is similar, except that the slope of the curve is now changed in sign. This condition is represented in figure (2).

It is the object of the present paper to study this "time distribution" of reactant gas at a definite point in the tube, and show how it can be expressed by the basic velocity equation.

Derivation of Equation for Time Distribution of Reactant Gas

In order to do this, we shall consider a definite point in the tube—for example, the outlet.

The equation previously developed[1] to express the velocity of diffusion controlled reactions of the type considered (in closed systems) is:

$$\frac{d(y_0^{1/3} - y^{1/3})}{d\theta} = \frac{Ap}{(y_0^{1/3} - y^{1/3}) + B}. \tag{1}$$

In this equation, θ refers to the time, p to the partial pressure of reactant gas, and y and y_0 are the absorption capacities of the solid at time θ and θ_0, respectively. It is easily seen that this equation is equally applicable to flow systems, and will express the reaction velocity at any point in the tube. However, unless we can express the variable y in terms of the pressure p, our task is hopeless.

This can be done by making use of the equation of continuity for intermittent extraction.[2] Expressed for our case, this equation may be written as

$$u\frac{\partial p}{\partial x} + \alpha\frac{\partial p}{\partial \theta} + \beta\frac{\partial z}{\partial \theta} = 0. \tag{2}$$

Here p and θ are, as above, the partial pressure of reactant gas, and the time, respectively; z refers to the amount of solid reaction product

[1] Hitchcock and Robinson, "Differential Equations in Applied Chemistry," 1936, 2nd. ed., p. 89.

formed; and u, α, and β, are constants. For the case in hand, where we are considering conditions at a definite point in the tube, x is constant, and equation (2) becomes

$$\frac{\partial p}{\partial \theta} = \left(-\frac{\beta}{\alpha}\right)\frac{\partial z}{\partial \theta}, \tag{3}$$

or

$$\partial p = \left(-\frac{\beta}{\alpha}\right)\partial z \tag{4}$$

Integrated, with the boundary conditions that $p = 0$ when $z = 0$, equation (4) becomes

$$z = \left(-\frac{\alpha}{\beta}\right)p, \quad \text{or}$$

$$z = ap, \quad \text{where } a = \left(-\frac{\alpha}{\beta}\right).$$

Since $(y_0 - y)$, the amount of absorption capacity used up, equals z, the amount of solid reactant formed, we may write

$$(y_0 - y) = ap. \tag{5}$$

Now, since $y_0 = z_\infty$, and $z_\infty = ap_\infty$, we have

$$y_0 = ap_\infty.$$

Substituting in (5), we get the value of y in terms of p:

$$y = a(p_\infty - p).$$

Applying this value of y to equation (1), the result is

$$\frac{d[(ap_\infty)^{1/3} - (a[p_\infty - p])^{1/3}]}{d\theta} = \frac{Ap}{[(ap_\infty)^{1/3} - (a[p_\infty - p])^{1/3}] + B}. \tag{6}$$

The constant B, as was shown in the previous paper (1), is representative of a *constant* diffusional resistance on the surface of the solid granules. The quantity

$$[(ap_\infty)^{1/3} - (a[p_\infty - p])^{1/3}]$$

is representative of the *variable* resistance to diffusion caused by the formation of the solid reaction product. One or the other of these quantities may become controlling, depending upon the type of reaction we are considering. In some cases, as has been shown previously,[1]

both quantities may be of importance. The equation may thus be integrated for three cases:

1. The reaction is controlled by a constant diffusional resistance;
2. It is controlled by the diffusional resistance of the reaction product;
3. It is controlled by a combination of the above, where both factors must be considered.

In this paper we shall consider only the case where the reaction is controlled by the resistance offered to the diffusion of the reactant gas by the solid reaction product, which seems to be the most probable mechanism. With this in mind, we get, for the reaction product-controlled case:

$$[(p_\infty)^{1/3} - (p_\infty - p)^{1/3}]^2 = \int_{\theta_0}^{\theta} \frac{2A}{a^{2/3}} p \, d\theta \qquad (7)$$

The integral on the right hand side may be evaluated graphically, as will be shown later.

Application of Time Distribution Equation to the Reaction Between Hydrogen Sulphide and Iron Oxide

We may now apply this equation to a complicated case, the reaction between hydrogen sulphide and ferric oxide. (This is the reaction that is of importance in the "dry purification" of city gas.) Some of the complicating factors in the application of the equation to available data on this reaction are:

1. The gas usually contains some oxygen, which reacts consecutively with the complex iron sulphide first formed;
2. We have here, not one solid reaction product, but a solid and a liquid one, since water is a product of the reaction.

As is quite generally known, the reaction of the iron sulphide with oxygen is slow compared to the reaction of the ferric oxide with hydrogen sulphide, and for the purposes of this illustration, the hydrogen sulphide-ferric oxide reaction may be taken as controlling. As will be shown, although we have a liquid reaction product, it is apparently held firmly enough by the solid reaction product, so that the solid and liquid together act as a solid phase.

Figure (3) shows the results of two runs, by different workers, in which gas containing 400 grains of hydrogen sulphide per 100 cu. ft. of gas was passed at constant rate through a tube containing a mixture of a by-product iron oxide and sawdust. The concentrations of hy-

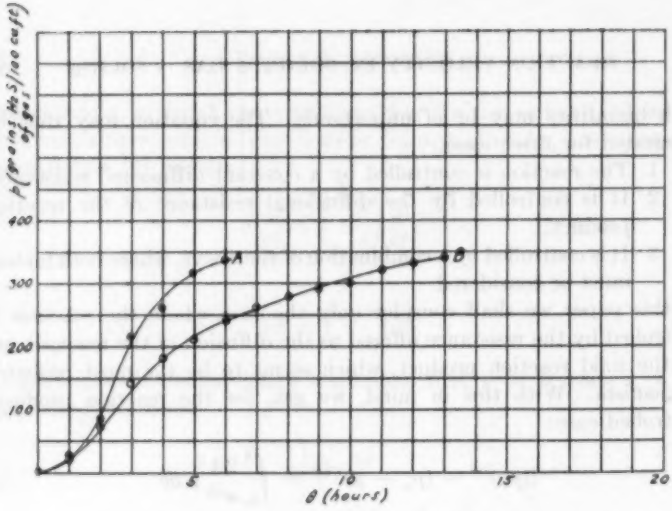

FIG. 3

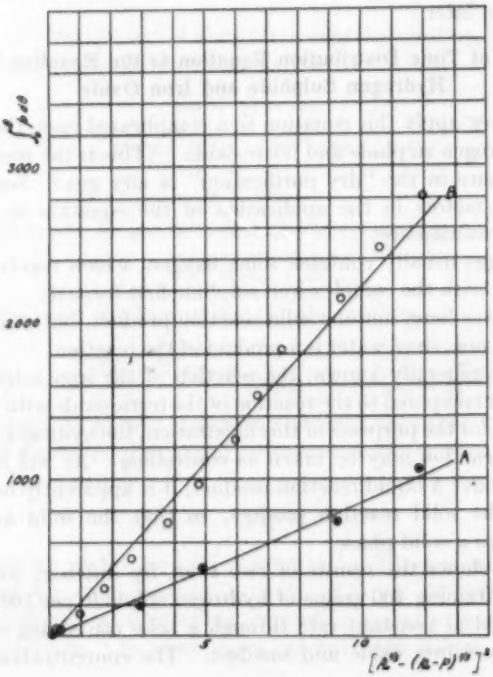

FIG. 4

drogen sulphide drop off with time in a manner similar to that in the curve shown in figure (2). (For the study of these results it is not necessary to convert the concentrations given to partial pressures, as the shape of the curve does not depend upon the units employed.) Curve A, figure (3), represents a run made in this laboratory by Dr. J. E. Brewer, and curve B, figure (3), represents a run made by Seil, Heiligman, and Clark.[3]

From a plot of concentration against time, similar to figure (3), but on a slightly different scale, we may determine the areas under the curve, from θ_0 to various values of θ, which will give us values proportional to the right hand side of equation (7). We may then plot these values against the calculated values of the left hand side of equation (7). This has been done for the data given in figure (3). The results are plotted in figure (4). As can be seen, a good value is obtained for the straight line in both cases.

Discussion of Results

The results on the hydrogen sulphide-iron oxide reaction are of particular interest, because they show that a reaction can depart considerably from ideal conditions, and still follow closely the velocity equation (6). A number of experimental results on this reaction have been plotted, (not shown here), and satisfactory agreement with the equation has been observed.

In the previous paper in this series (1) the discontinuity of equation (1) has been discussed. In this case, also, it will be found that the straight line will not be prolonged indefinitely, since as was previously shown, absorption takes place in *finite time*. The curve in figure (4) should therefore break off sharply, and approach infinity parallel to the axis of $\int_{\theta_0}^{\theta} p\, d\theta$. The sharpness with which this occurs is a measure of the approach of the experimental results to theoretical conditions.

It should be emphasized in all of this work, that our basic equations are applicable to conditions in which a *second solid phase* is a product of the reaction, and not a *solid solution*, since we have postulated that the reaction takes place at the boundary of the two solid phases. As to whether the equation would hold for adsorption of vapors by charcoal, we are not in a position to say, but believe that it would not, as this is probably a solid solution process. At any rate, it does give us a method for deciding between the two types of solid-gas reactions.

[3] Seil, Heiligman, and Clark, American Gas Journal, May and June, 1929.

The writer wishes to express his appreciation to Drs. Frank L. Hitchcock, of Massachusetts Institute of Technology, E. R. Weaver of the Bureau of Standards, and J. Edward Brewer, of this laboratory, for reading the manuscript, and to the latter for permission to reproduce the results given in curve A, figure (3).

Summary

1. A velocity equation, previously developed to express the reaction velocity in closed systems between a gas and a solid, for the case in which the reaction product is a solid phase, and in which the actual reaction rate *at* the reacting surface is great compared to the rate of supply of the reactant gas *to* the reacting surface, has been applied to flow systems.

2. The equation expressing the change of partial pressure of reactant gas with time at any point in a tube in which such a flow reaction is taking place has been found to be:

$$\frac{d[(ap_\infty)^{1/3} - (a[p_\infty - p])^{1/3}]}{d\theta} = \frac{Ap}{[(ap_\infty)^{1/3} - (a[p_\infty - p])^{1/3}] + B}$$

where p refers to partial pressure of reactant gas, θ to the time, and a, A and B are constants.

3. This equation has been applied to the reaction between hydrogen sulphide and iron oxide in flow systems. Satisfactory agreement has been observed between theory and experiment.

Laboratory of Brewer and Gardner, Philadelphia.

R-MATRICES AND EQUIVALENT NETWORKS. I[1]

By Richard Stevens Burington

1. Introduction. In recent papers[2] the author has considered a theory relating to m-affine congruent matrices and their application to the study of electrical networks. If B is a matrix of a linear passive m-terminal pair network and T is any real m-affine non-singular matrix, then $A = T^\tau BT$ corresponds to a network *equivalent* to B, that is, equivalent in the electrical sense that A and B have the same characteristic coefficient matrices. From physical considerations it is desirable to be able to determine the ranges V of the elements in T so that A corresponds to networks which do not require the introduction of one or more ideal transformers for realizability.[3]

In part I of the present paper the author initiates the development of a theory relating to R-matrices. Necessary and sufficient conditions are given that $C = P^\tau DP$ be an R-matrix, where P and D are real and P non-singular. With each D there is associated n sets of functions $\chi^{(i)}$, each set forming a group of order 2^{n-1}. If C is positive definite then the $n \cdot 2^{n-1}$ quadric surfaces $\chi^{(i)} = 0$ form the boundary of a region W in the space of P for which C is an R-matrix. The theory is then extended to sets of real matrices and to the case when P is m-affine.

In the second part of the paper application of the theory of R-matrices is made to linear electrical networks, sufficient conditions being found, together with a range V of T, that networks corresponding to A not require ideal transformers for realizability.

Part I. R-matrices.

2. R-matrices. *A real matrix $B = (b_{rs})$ of order n is said to be an R-matrix, if and only if $\sum_{r=1}^{n} |b_{rs}| \leqq 2b_{ss}$, for each $s = 1, 2, \cdots, n$.*

[1] Presented to the American Mathematical Society, Cambridge Mass., Sept. 2, 1936.

[2] Burington, Richard S., *On the equivalence of quadrics in m-affine n-space and its relation to the equivalence of 2m-pole networks.* Transactions of Amer. Math. Soc., Vol. 38 No. 1, July 1935, pp. 163–176.

Burington, Richard S., *Matrices in electric circuit theory.* Journal of Mathematics and Physics, Vol. XIV, No. 4, Dec. 1935.

[3] Guillemin, E. A., *Communication Networks.* Vol. 2, (1935), John Wiley and Sons, New York, p. 151, pp. 232–234.

If B is a real R-matrix, then $b_{ss} \geqq 0$, $s = 1, 2, \cdots, n$. If some $b_{ss} < 0$, then evidently B cannot be an R-matrix, so that *a necessary condition that B be an R-matrix is that* $b_{ss} \geqq 0$, $s = 1, \cdots, n$.

Let A be a real matrix congruent to B,

$$A = T^T B T, \tag{1}$$

where B is a positive semi-definite real matrix, and where $T = (t_{rs})$ is real and non-singular. If B is of rank ρ, then so is A, for T is non-singular.[4]

In order that A be an R-matrix, it is necessary that the elements of T be such that $\sum_{r=1}^{n} |a_{rs}| \leqq 2a_{ss}$, $s = 1, 2, \cdots, n$. We wish to determine the range of the elements of T in order that A be an R-matrix. Since B is positive semi-definite, so is A, and consequently we can assert that $a_{ss} \geqq 0$ as well as $b_{ss} \geqq 0$, $s = 1, 2, \cdots, n$.

Let μ and δ be operators such that if c_{rs} is the element in the r^{th} row and s^{th} column of a $m \times n$ array $C = (c_{rs})$, that

$$\mu C_{rs} = -c_{rs}, \qquad \delta C_{rs} = -|c_{rs}| \text{ if } r \neq s;$$

$$\mu C_{rr} = c_{rr}, \qquad \delta C_{rr} = c_{rr}.$$

If $(x) = (x_1, \cdots, x_m)$ be a row "vector", then so is $(x)C = (y)$, where $(y) = (y_1, \cdots, y_m)$, and $y_k = \sum_{j=1}^{m} x_j c_{jk}$, for $k = 1, 2, \cdots, n$.

Hence by letting $x_j = \mu$, and δ in turn, for $j = 1, \cdots, m$, we see that

$$(\mu)C = (c_{11} - c_{21} - \cdots - c_{m1}, \cdots, -c_{1k} - c_{2k} - \cdots - c_{k-1,k} + c_{kk} - c_{k+1,k} - \cdots - c_{mk}, \cdots, -c_{1n} - \cdots - c_{mn})$$

$$(\delta)C = (c_{11} - |c_{21}| - \cdots - |c_{m1}|, \cdots, -|c_{1k}| - |c_{2k}| - \cdots - |c_{k-1,k}| + c_{kk} - |c_{k+1,k}| - \cdots - |c_{mk}|, \cdots, -|c_{1n}| - \cdots - |c_{mn}|).$$

Evidently, $(\mu)C \geqq (\delta)C$. If $(\delta)C \geqq 0$, so is $(\mu)C \geqq 0$.

From the definition of an R-matrix, it follows that a real matrix B is an R-matrix if and only if $(\delta)B \geqq (0)$.

Consider the set of all possible distinct vectors of the form $(x_q) = (x_{q1}, \cdots, x_{qn})$, in which the x_{qj} are either 1 or -1, $j = 1, 2, \cdots, n$. There are 2^n such vectors $(x_1), \cdots, (x_{2^n})$.

Let

$$\chi_q = (x_q)A \qquad (q = 1, \cdots, 2^n)$$

[4] C. C. MacDuffee, *Theory of Matrices*, pp. 57–8.

be the set of functions generated from the real matrix A and the vectors (x_q).

With each χ_q we associate the diagonal matrix

$$[X_q] \equiv [x_{q1}, \cdots, x_{qn}].$$

It is easy to prove that the set of matrices $[X_q]$, $q = 1, \cdots, 2^n$, form a group G of order 2^n under the operation of multiplication. The generators of G are $[-1, 1, \cdots, 1]$, $[1, -1, \cdots, 1]$, $[1, 1, -1, 1, \cdots, 1]$, $\cdots$, $[1, 1, \cdots, 1, -1]$.

We shall be interested in certain subgroups $G^{(k)}$, $k = 1, \cdots, n$, of G, where in $G^{(k)}$, $x_{qk} = x_{qk}^{(k)} = 1$. The order of each $G^{(k)}$ is 2^{n-1}, $k = 1, \cdots, n$.

Let $(x_q^{(k)})$, $q = 1, \cdots, 2^{n-1}$, denote the sub-sets of (x_q), $q = 1, \cdots, 2^n$, corresponding to $G^{(k)}$ and let $\chi_q^{(k)}$, $q = 1, \cdots, 2^{n-1}$ be the set of functions appearing in the k^{th} column of $X^{(k)}A$, where $X^{(k)}$ is a rectangular array with $(x_q^{(k)})$ in the q^{th} row.

Theorem 2.1. *If A be a matrix then the function $\delta^{(k)}A$ is an invariant under any transformation of the group $G^{(k)}$, where $\delta^{(k)}A$ is the k^{th} component of the vector $(\delta)A = (\delta^{(1)}A, \cdots, \delta^{(n)}A)$, $(k = 1, \cdots, n)$.*

The proof of this theorem is immediate, since the various $\chi_q^{(k)}$, $q = 1, \cdots, 2^{n-1}$ differ only in the signs preceding the various a_{rs}.

Similarly,

Theorem 2.2. *The set of functions χ_q, form a group g under the substitutions implied by (x_q), and g is simply isomorphic with G, $(q = 1, \cdots, 2^n)$.*

Theorem 2.3. *The set of functions $\chi_q^{(k)}$, $(q = 1, \cdots, 2^{n-1})$, form a subgroup $g^{(k)}$ of g simply isomorphic with $G^{(k)}$, and the function $\delta^{(k)}A$ is an invariant of $g^{(k)}$ as well as of $G^{(k)}$, $(k = 1, \cdots, n)$.*

It is clear that every member of $\chi_q^{(k)}$, $q = 1, \cdots, 2^{n-1}$ can be generated from one of the set by a generator of the group $g^{(k)}$. Consequently all the functions $\chi_q^{(k)}$, $q = 1, \cdots, 2^{n-1}$; $k = 1, 2, \cdots, n$, can be generated from $(\mu)A$.

If, for a given k, $\chi_q^{(k)} \geqq 0$, $(q = 1, \cdots, 2^{n-1})$, that is if each member of $g^{(k)}$ is non-negative, then $\delta^{(k)}A \geqq 0$. Furthermore, if this is true for each value of $k = 1, \cdots, n$, then $(\delta)A \geqq (0)$.

Theorem 2.4. *A necessary and sufficient condition that the real matrix $A = T^TBT$, $d(T) \neq 0$, be an R-matrix is that the members of each group $g^{(k)}$ be all non-negative.*

3. The surfaces $\chi_q^{(k)} = 0$. Let $\mathcal{T}$ be an n^2-space consisting of the totality of real points $(t_{11}, \cdots, t_{n1}; \cdots; t_{1n}, \cdots, t_{nn})$, that is, the

space of the matrix T. In general, each member χ_q of g is a quadric in $2n$ variables imbedded in $\mathfrak{T}$.

The various surfaces $\chi_q^{(k)} = 0, q = 1, \cdots, 2^{n-1}$, in $g^{(k)}$, k fixed, divide $\mathfrak{T}$ into regions $\pi^{(k)}$ and $\eta^{(k)}$, where $\pi^{(k)}$ contains all points of $\mathfrak{T}$ for which $\chi_q^{(k)} \geqq 0$, $q = 1, \cdots, 2^{n-1}$, and where $\eta^{(k)}$ contains all other points of $\mathfrak{T}$ not in $\pi^{(k)}$. In other words, in $\pi^{(k)}$, $\delta^{(k)}A \geqq 0$.

Hence with each group $g^{(k)}$, there is associated a region $\pi^{(k)}$ for which $\delta^{(k)}A \geqq 0$, and a region for which $\delta^{(k)}A < 0$, $k = 1, \cdots, n$.

Let π denote that portion of $\mathfrak{T}$ composed of all points common to $\pi^{(1)}, \cdots, \pi^{(n)}$, and no other points. Then we have, *a necessary and sufficient condition that* $(\delta)A \geqq (0)$ *is that the elements of* T *belong to* π.

Delete from the region π those points in $\mathfrak{T}$ for which $d(T) = 0$. Denote this region by W. Then

Theorem 3.1. *A necessary and sufficient condition that the matrix* $A = T^TBT$ *be an* R*-matrix, where* B *is real and* T *is real and non-singular, is that the elements of* T *belong to the region* W *in* $\mathfrak{T}$.

Let B be a positive semi-definite real matrix. Then every $a_{ss} \geqq 0$, $s = 1, \cdots, n$, in T^TBT. It is well known that there exists a real transformation matrix T_1 such that $J = T_1^TBT_1$, where J is the matrix $[1, \cdots, 1, 0, \cdots, 0]$.

Evidently J is an R-matrix.

Consider the matrix A_1 congruent to J, where $T_2 = [1, \cdots, t, 0, \cdots, 0]$, $t \neq 0$. Then $A_1 = T_2^TJT_2 = [1, \cdots, t^2, 0, \cdots, 0]$, $t \neq 0$. Obviously A_1 is an R-matrix for all real values of t. Now $A_1 = T_2^TT_1^TBT_1T_2$. Hence there exists a region W containing an infinite number of points for which A_1 is an R-matrix.

Theorem 3.2. *If* B *be a positive semi-definite real matrix, there exists an infinite number of points in* $\mathfrak{T}$, *for which* $A = T^TBT$ *is an* R*-matrix. Here* B *need not be an* R*-matrix.*

Every point on the boundary of the region W given in theorem 3.1, except possibly those points where $d(T) = 0$, is a point for which A is a positive semi-definite R-matrix.

Example 1. Suppose

$$B = \begin{pmatrix} 20, & 4 \\ 4, & 1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1, & 0 \\ x, & y \end{pmatrix}, \quad y \neq 0.$$

Then

$$A = T^TBT = \begin{pmatrix} 1, & x \\ 0, & y \end{pmatrix}\begin{pmatrix} 20, & 4 \\ 4, & 1 \end{pmatrix}\begin{pmatrix} 1, & 0 \\ x, & y \end{pmatrix} = \begin{pmatrix} 20 + 8x + x^2, & (4 + x)y \\ (4 + x)y, & y^2 \end{pmatrix}.$$

B is not an R-matrix. We seek the region W for which A is an R-matrix. Evidently, $a_{11} = 20 + 8x + x^2$ and $a_{22} = y^2$ are non-negative. Now

$$(\delta)A = (\delta^{(1)}A, \delta^{(2)}A) = ((20 + 8x + x^2) - |y(4 + x)|, y^2 - |y(4 + x)|).$$

Following the theory developed above, we consider the group G of diagonal matrices $[X_q]$, $q = 1, \cdots, 2^2$; namely

$$[X_1] = [1, 1], \quad [X_2] = [1, -1], \quad [X_3] = [-1, 1], \quad [X_4] = [-1, -1].$$

Consider the subgroups $G^{(1)}$ and $G^{(2)}$ of G, where $G^{(1)}$ consists of $[X_1^{(1)}] = [X_1] = [1, 1]$ and $[X_2^{(1)}] = [X_2] = [1, -1]$, and where $G^{(2)}$ consists of $[X_1^{(2)}] = [X_1] = [1, 1]$ and $[X_2^{(2)}] = [X_3] = [-1, 1]$. Let $(x_1^{(1)})$ be the row vector $(1, 1)$ corresponding to the diagonal matrix $[X_1^{(1)}]$; $(x_2^{(1)})$ the row vector $(1, -1)$ corresponding to $[X_2^{(1)}]$; and $(x_1^{(2)})$ and $(x_2^{(2)})$ the row vectors $(1, 1)$ and $(-1, 1)$ corresponding to $[X_1^{(2)}]$ and $[X_2^{(2)}]$, respectively. Denote by $X^{(1)}$ and $X^{(2)}$ the arrays

$$X^{(1)} = (x_q^{(1)}) = \begin{pmatrix} 1, & 1 \\ 1, & -1 \end{pmatrix}, \qquad X^{(2)} = (x_q^{(2)}) = \begin{pmatrix} 1, & 1 \\ -1, & 1 \end{pmatrix}.$$

We let $\chi_q^{(k)}$ denote the set of functions appearing in the k^{th} column of $X^{(k)}A$; that is, $\chi_q^{(1)}$ is the set appearing in the first column of

$$X^{(1)}A = \begin{pmatrix} 1, & 1 \\ 1, & -1 \end{pmatrix}\begin{pmatrix} 20 + 8x + x^2, & (4 + x)y \\ (4 + x)y, & y^2 \end{pmatrix}$$

$$= \begin{pmatrix} 20 + 8x + x^2 + 4y + xy, & (4 + x)y + y^2 \\ 20 + 8x + x^2 - 4y - xy, & (4 + x)y - y^2 \end{pmatrix},$$

and $\chi_q^{(2)}$ is the set appearing in the second column of

$$X^{(2)}A = \begin{pmatrix} 1, & 1 \\ -1, & 1 \end{pmatrix}\begin{pmatrix} 20 + 8x + x^2, & (4 + x)y \\ (4 + x)y, & y^2 \end{pmatrix}$$

$$= \begin{pmatrix} 20 + 8x + x^2 + 4y + xy, & (4 + x)y + y^2 \\ -20 - 8x - x^2 + 4y + xy, & -(4 + x)y + y^2 \end{pmatrix}.$$

In order that A be an R-matrix, each member of the group $g^{(1)}$ of functions $\chi_1^{(1)} = 20 + 8x + x^2 + 4y + xy$, $\chi_2^{(1)} = 20 + 8x + x^2 - (4y + xy)$ and each member of the group $g^{(2)}$ of functions $\chi_1^{(2)} = (4 + x)y + y^2$, $\chi_2^{(2)} = -(4 + x)y + y^2$ must be non-negative. The various members of $g^{(1)}$ and $g^{(2)}$ are quadratic in the parameters x and y and the

various surfaces $\chi_1^{(1)} = 0$, $\chi_2^{(1)} = 0$, $\chi_1^{(2)} = 0$, $\chi_2^{(2)} = 0$ define a region π in the xy-plane for which $(\delta)A$, as given above, is non-negative. From π delete all points for which $y = 0$ and denote the resulting region by W. Then, a necessary and sufficient condition that A be an R-matrix is that (x, y) belong to W.

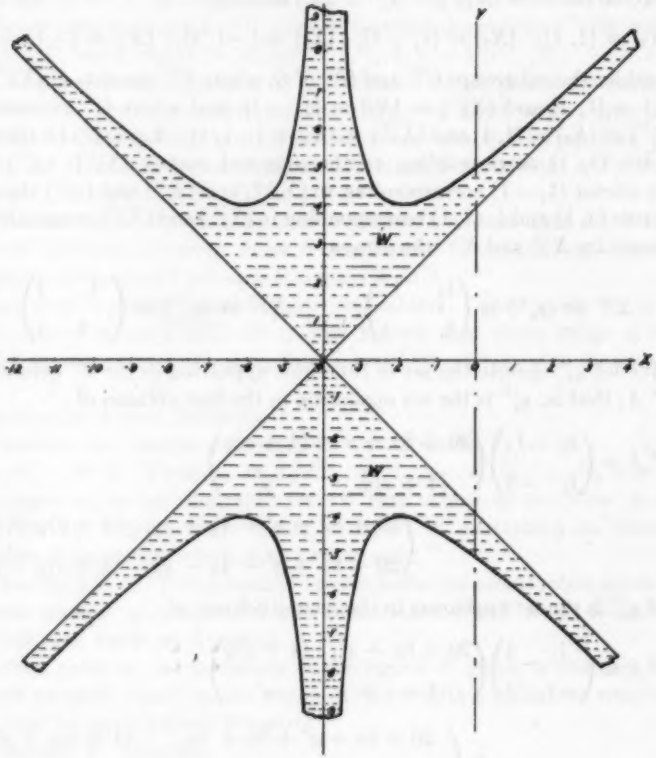

FIG. 1

The region W is clearly indicated in figure 1. All points on the boundary of W belong to W except the point $(-4, 0)$.

4. **Pairs of matrices.** In this section we shall consider pairs of real matrices B_1 and B_2, where B_1 is positive definite and B_2 is positive

semi-definite. It is known that there exists a real non-singular matrix T_1 which reduces B_1 and B_2 to the diagonal forms

$$(4.1)\quad I = [1, \cdots, 1] = T_1^T B_1 T_1,\ J = [\alpha_{11}, \cdots, \alpha_{rr}, 0, \cdots, 0] = T_1^T B_2 T_1,$$

where each $\alpha_{ii} \geqq 0$. Both I and J are R-matrices.

Let $T_2 = [1, \cdots, 1, t]$. Then I and J become

$$C_1 = [1, \cdots, t^2] = T_2^T I T_2,\ C_2 = [\alpha_{11}, \cdots, \alpha_{rr}, 0, \cdots, 0] = T_2^T J T_2.$$

Evidently, C_1 and C_2 are both R-matrices for all real values of t.

Let W_1 denote the region for which $A_1 = T^T B_1 T$ is an R-matrix and W_2 the region for which $A_2 = T^T B_2 T$ is an R-matrix. The set of all points W_{12} common to W_1 and W_2 constitutes a region for which A_1 and A_2 are R-matrices. Hence

Theorem 4.1. *If B_1 and B_2 are real matrices, B_1 being positive definite and B_2 positive semi-definite, there exists a region W_{12} in $\mathfrak{T}$ for which $A_1 = T^T B_1 T$, $A_2 = T^T B_2 T$ are both R-matrices. The region W_{12} contains infinitely many points.*

Consider the set of real positive semi-definite matrices $B_1, B_2, \cdots, B_q$. Let $W_1, \cdots, W_q$ be, respectively, the regions for which $A_1, \cdots, A_q$ are R-matrices, where $A_j = T^T B_j T$, $j = 1, \cdots, q$. Let $W_{1,\cdots,q}$ denote the set of points (if any) common to $W_1, \cdots, W_q$. From previous considerations we see that a necessary and sufficient condition that the positive semi-definite real matrices $A_j = T^T B_j T$, $j = 1, \cdots, q$, be simultaneously R-matrices is that $W_{1,\cdots,q}$ be non-vacuous, and that T belong to $W_{1,\cdots,q}$.

Evidently, if $B_1, \cdots, B_q$ are each R-matrices the region $W_{1,\cdots,q}$ is non-vacuous for $T = [1, \cdots, 1]$ belongs to $W_{1,\cdots,q}$.

The various theorems of §2 and §3 may be extended, with suitable modifications, to cover sets of matrices $A_1, \cdots, A_q$.

5. *T* ***m*-affine.** Consider the case where T is a real 1-affine transformation of the form

$$(5.1)\quad \begin{pmatrix} 1 & 0 & \cdots & 0 \\ t_{21} & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & & \vdots \\ t_{n1} & t_{n2} & & t_{nn} \end{pmatrix},\quad d(T) \neq 0.$$

In a previous paper[5] the author showed that a real non-parabolic matrix B can be reduced by (5.1) to the form

[5] Trans., loc. cit.[2]

$$A_1 = \begin{pmatrix} \alpha_{11} & 0 & 0 \cdots & 0 & 0 \\ 0 & \delta_2 & 0 \cdots & 0 & 0 \\ 0 & 0 & \delta_3 \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdots & \delta_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & \delta_n \end{pmatrix}, \tag{5.2}$$

where α_{11} is an absolute invariant and the δ's are 1, -1 or 0.

Clearly, if $\alpha_{11} < 0$, A_1 is not an R-matrix.

We shall assume that B is positive semi-definite, so that $\alpha_{11} > 0$, and δ_j are either 1 or 0. As in §3 we may prove that

Theorem 5.1. *Theorems* 2.1 *through* 2.4 *and theorems* 3.1 *and* 3.2 *hold with* T *1-affine, and* B *positive semi-definite, non-parabolic.*

Suppose that T is 2-affine, that is

$$T = \begin{pmatrix} 1 & 0 \cdots & 0 \\ 0 & 1 \cdots & 0 \\ t_{21} & t_{22} \cdots & t_{2n} \\ \vdots & & \vdots \\ t_{n1} & \cdots & t_{nn} \end{pmatrix}, \quad d(T) \neq 0. \tag{5.3}$$

If B is a real non-parabolic positive definite symmetric matrix it can be reduced by a matrix T of (5.3) to the form

$$A_1 = \left(\begin{array}{cc|ccc} \alpha_{11} & \alpha_{12} & 0 & \cdots & 0 \\ \alpha_{21} & \alpha_{22} & 0 & \cdots & 0 \\ \hline 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{array}\right) = T_1^T B T_1. \tag{5.4}$$

Here the α_{11}, α_{12}, α_{22} are absolute invariants. Clearly A_1 is not necessarily an R-matrix. If A_1 is not an R-matrix, no transformation (5.3) will reduce (5.4) to an R-matrix and maintain the essential form of (5.4).

Theorem 5.2. *If* B *is real symmetric non-parabolic and positive semi-definite, and if* A_1 *given in* (5.4) *is an* R*-matrix, then there exists a non-vacuous region* W *for which* $T^T B T$ *is an* R*-matrix,* W *contains infinitely many points.*

Similar theorems may be proven for T m-affine, $m > 2$.

In general, with suitable modifications the theorems of §2, §3 and §4 may be extended to the case where T is m-affine.

The detailed considerations of the case where B is parabolic and not positive semi-definite as well as the extension to sets of matrices will be considered at a later time.

If further restrictions be placed on T, then the region W will be restricted accordingly. For example, if in $A = T^T B T$ it be furthermore required that $a_{r_1 s_1} = 0, \cdots, a_{r_k s_k} = 0$, then the points of T for which A is an R-matrix consists of those points in W, if any, laying on the intersections of the surfaces $a_{r_1 s_1} = 0, \cdots, a_{r_k s_k} = 0$.

Part II. Application to network theory[6]

6. **Network equations.** We shall consider the m-terminal pair n-mesh linear electrical network containing (lumped) resistances, inductances and elastances.

The complete differential equations for this network are

$$A\{i\} = \{e\}, \tag{6.1}$$

where

$$a_{st} = R_{st} + L_{st}p + D_{st}p^{-1}, \qquad p = \frac{d}{dt},$$

$\{i\} = \{i_1, \cdots, i_n\}$ and $\{e\} = \{e_1, \cdots, e_m, 0, \cdots, 0\}$ are one-column arrays. Here $e_1, \cdots, e_m$ are the real e.m.f.'s impressed on terminal pairs $1, \cdots, m$, respectively; q_s and i_s, are the instantaneous (real) charges and (real) currents, respectively, in mesh s; R_{st}, L_{st}, D_{st}, $s \neq t$, (real numbers) are the (lumped) circuit parameters (resistance, inductance and elastance, respectively) mutual to meshes s and t; R_{ss}, L_{ss}, D_{ss}, the total circuit (real) parameters of mesh s, i.e., the total resistance, inductance and elastance of mesh s, $(s, t = 1, \cdots, n)$. The meshes are so choosen that mesh s is the only one which passes through the terminal pair s, $(s = 1, \cdots, m)$.

Matrix $A = (a_{rs})$ is called a *network* matrix. We shall assume that the mutual impedances a_{rs} are (bilateral) reciprocal; i.e., that A is symmetric.

Let the e.m.f.'s be given by

$$\{e\} = \{E\}e^{j\omega t} \tag{6.2}$$

[6] Burington, R. S., *Matrices in electric circuit theory*, Journal of Mathematics and Physics, Vol. XIV, No. 4, Dec. 1935, pp. 336–344.

where $\{e\} = \{e_1, \cdots, e_m, 0, \cdots, 0\}$ and $\{E\} = \{E_1, \cdots, E_m, 0, \cdots, 0\}$ are column vectors, E_s is the complex-root-mean square voltage amplitude for pair s, and the real part of e_s is the actual e.m.f., ϵ_s, in mesh s. Then the *steady-state-solution* of (6.1) is (*if* $d(A) \neq 0$)

$$\{I\} = B^{-1}\{E\}, \tag{6.3}$$

where

$$B = (b_{rs}), \quad b_{rs} = b_{sr} = R_{rs} + L_{rs}\lambda + D_{rs}\lambda^{-1}, \quad \lambda = j\omega. \tag{6.4}$$

7. **Equivalent networks.** If $\{I\}_m$ and $\{E\}_m$ denote $\{I\}$ and $\{E\}$, respectively, with all but the first m rows deleted, and $Y = (Y_{st})$ denotes B^{-1} with all but the first m rows and columns deleted, then

$$\{I\}_m = Y\{E\}_m. \tag{7.1}$$

Y is known as a *characteristic coefficient* (*admittance*) *matrix* of network A.

Two $2m$-terminal linear networks, of matrices A and B, respectively, are said to be *equivalent* if for all frequencies ($\omega = -\lambda j$), they have equal characteristic coefficient matrices Y.

The rate at which energy is being supplied to the network (6.1), the instantaneous power, is

$$\beta = (i)\{\epsilon\} = (i)A\{i\} = (i)R\{i\} + p(i)L\{i\}/2 + p(q)D\{q\}/2. \tag{7.2}$$

Let β be subjected to the real cogredient m-affine non-singular linear transformations

$$\{i\} = T\{\bar{i}\}, \qquad \{q\} = T\{\bar{q}\}. \tag{7.3}$$

Then β becomes

$$(\bar{i})\bar{A}\{\bar{i}\} = (\bar{i})\{\epsilon\},$$

where

$$\bar{A} = T^T A T.$$

Hence

$$\bar{B} = T^T B T, \tag{7.4}$$

where

$$\bar{B} = \bar{R} + \bar{L}\lambda + \bar{D}\lambda^{-1}, \qquad \lambda = j\omega. \tag{7.5}$$

It has been shown that *the network of matrices* B *and* $\bar{B}$ *are equivalent.* Moreover, if B is passive (i.e., R, L and D are non-negative) then $\bar{B}$ is passive.

However, even though B corresponds to a physically realizable network not containing an ideal transformer,[7] the network corresponding to $\bar{B}$ may not be realizable except by means of ideal transformers. The network corresponding to $\bar{B}$ will require one or more ideal transformers whenever $\bar{B}$ contains *negative circuit elements.*

Such negative circuit elements may occur whenever—for all possible mesh contour selections—the total resistance (inductance or elastance) on a mesh contour is less than the numerical value of the sum of the various resistances (inductances or elastances) in the various mutual branches of the mesh.

Networks having an undue number of ideal transformers are of little practical value at present, hence we wish to avoid ideal transformers whenever possible. Consequently, transformations $T = (t_{rs})$ which lead to *negative circuit elements* (resistances, inductances and capacitances) in $\bar{B}$ of (7.5) are impractical.

Evidently, the theory of R-matrices is applicable.

The various mathematical theorems presented in part I of this paper may be used. For example, by §3, 4, 5,

Theorem 7.1. *If B is the matrix of a given 1-terminal pair network which involves only one type of circuit parameter, (resistance, inductance or elastance) and T is 1-affine and non-singular, then a sufficient condition that $A = T^tBT$ correspond to a network $\mathfrak{A}$ equivalent to the network of matrix B, but $\mathfrak{A}$ not requiring one or more ideal transformers for realizability, is that A be an R-matrix that is, that T belong to a region W.*

Theorem 7.2. *If $A = T^tBT$ be the matrix of a network $\mathfrak{A}$ equivalent to a network $\mathfrak{B}$ of matrix B, and $\mathfrak{B}$ is a 1-terminal pair network, T 1-affine and non-singular, $\mathfrak{B}$ and $\mathfrak{A}$ each containing but two types of circuit parameters, say inductance L and elastance D, $B = \lambda L + \lambda^{-1}D$, then a sufficient condition that $\mathfrak{A}$ not require an ideal transformer for realizability is that $\bar{L}$ and $\bar{D}$ each be an R-matrix, where $A = \lambda\bar{L} + \lambda^{-1}\bar{D}$, that is that T belong to the region W_{12} common to the regions W_1 and W_2 for which $\bar{L}$ and $\bar{D}$ are R-matrices, respectively.*

Theorem 7.3. *In theorem 7.2, if $B = \lambda L + \lambda^{-1}D + R$, that is, if $\mathfrak{B}$ contains three types of circuit parameters, then a sufficient condition that $\mathfrak{A}$, equivalent to $\mathfrak{B}$, require no ideal transformers for realizability, is that $\bar{L}$, $\bar{D}$, $\bar{R}$, all be R-matrices, $(A = \lambda\bar{L} + \lambda^{-1}\bar{D} + \bar{R})$, that is T belong to region W_{123}, the region common to regions W_1, W_2, W_3 for which $\bar{R}$, $\bar{L}$, $\bar{D}$ are R-matrices, respectively; and that W_{123} be non-vacuous.*

[7] Guillemin, loc. cit.[3]

Cauer, W., *Ideal Transformatoren und lineare Transformationen*, E.N.T. 9 (1932) s. 157.

W_{123} contains at least one point when B is the matrix of a network not requiring ideal transformers for realizability.

Similar theorems can be stated in regard to sufficient conditions for the realizability of networks requiring ideal transformers. Other theorems concerning the groups of surfaces $\chi_q^{(k)} = 0$ can also be stated.

Example 2. Let

$$R = \begin{pmatrix} 6, & -4 \\ -4, & 7 \end{pmatrix}$$

be the resistance matrix of a 1-terminal pair passive realizable resistance network of two meshes. Then if

$$T = \begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix}, \quad y \neq 0,$$

$A = T^T R T$ is the matrix of a resistance network equivalent to the network $\mathfrak{B}$ of matrix R, and

$$A = \begin{pmatrix} 1 & x \\ 0 & y \end{pmatrix}\begin{pmatrix} 6, & -4 \\ -4, & 7 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix}$$

$$= \begin{pmatrix} 6 - 8x + 7x^2, & y(-4 + 7x) \\ y(-4 + 7x), & 7y^2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

In order that A be the matrix of a resistance network realizable without the aid of ideal transformers, A must be an R-matrix, that is, $(\delta)A \geqq 0$. The curves which define the boundary of the region W of the xy-plane for which A is an R-matrix are, with $y \neq 0$,

$$\chi_1^{(1)} \equiv (6 - 8x + 7x^2) + y(-4 + 7x) = 0,$$

$$\chi_2^{(1)} \equiv (6 - 8x + 7x^2) - y(-4 + 7x) = 0,$$

$$\chi_1^{(2)} \equiv y(-4 + 7x) + 7y^2 = 0,$$

$$\chi_2^{(2)} \equiv -y(-4 + 7x) + 7y^2 = 0.$$

All points (x, y) interior to and on the boundary of W, except where $y = 0$, belong to W. W is clearly indicated in figure 2. If p be any point (x, y) in W, then A is the matrix of a resistance network $\mathfrak{A}$ equivalent to $\mathfrak{B}$ and realizable without ideal transformers. There are no other values of (x, y) for which $\mathfrak{A}$ is realizable without the aid of ideal transformers.

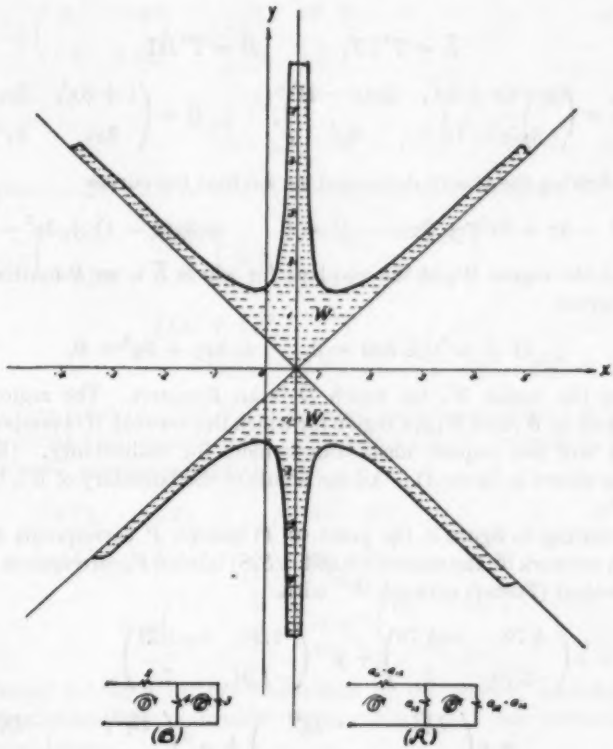

FIG. 2

Example 3. Consider the two-terminal reactive network $\mathfrak{B}^3$ with $L_1 = 1, L_3 = 3, D_1 = 1, D_3 = 3$. The network matrix B of $\mathfrak{B}$ is

$$B = p\begin{pmatrix} 4, & -3 \\ -3, & 3 \end{pmatrix} + p^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = pL + p^{-1}D.$$

Consider the matrices

$$A = T^tBT = p\bar{L} + p^{-1}\bar{D}$$

where

$$T = \begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix}, \quad y \neq 0.$$

[3] Guillemin, loc. cit.[2], pp. 236–8.

Then

$$\bar{L} = T^T L T, \qquad \bar{D} = T^T D T$$

$$\bar{L} \equiv \begin{pmatrix} 4 - 6x + 3x^2, & 3y(x-1) \\ 3y(x-1), & 3y^2 \end{pmatrix}, \qquad \bar{D} \equiv \begin{pmatrix} 1 + 3x^2, & 3xy \\ 3xy, & 3y^2 \end{pmatrix}.$$

Following the theory developed we see that the curves

$$(4 - 6x + 3x^2) \pm 3y(x-1) = 0, \qquad \pm 3y(x-1) + 3y^2 = 0,$$

define the region W_1 of the xy-plane for which $\bar{L}$ is an R-matrix; and the curves

$$(1 + 3x^2) \pm 3xy = 0, \qquad \pm 3xy + 3y^2 = 0,$$

define the region W_2 for which $\bar{D}$ is an R-matrix. The region W_{12} common to W_1 and W_2 is a region for which the network $\mathfrak{C}$ corresponding to A will not require ideal transformers for realizability. (Region W_{12} is shown in figure 3). All the points of the boundary of W_{12} belong to W_{12}.

Referring to figure 3, the point (0, 1) labeled P corresponds to the given network $\mathfrak{B}$; the point $(-0.264, 1.528)$ labeled P_2 corresponds to the equivalent (Foster) network $\mathfrak{B}^{(2)}$ with

$$A^{(2)} = p \begin{pmatrix} 5.79, & -5.79 \\ -5.79, & 7 \end{pmatrix} + p^{-1} \begin{pmatrix} 1.21, & -1.21 \\ -1.21, & 7 \end{pmatrix}$$

$$\equiv p \begin{pmatrix} L_2^{(2)}, & -L_2^{(2)} \\ -L_2^{(2)}, & L_2^{(2)} + L_4^{(2)} \end{pmatrix} + p^{-1} \begin{pmatrix} D_2^{(2)}, & -D_2^{(2)} \\ -D_2^{(2)}, & D_2^{(2)} + D_4^{(2)} \end{pmatrix};$$

the point $(1, -1.33)$ labeled P_3 corresponds to the (Cauer) equivalent network $\mathfrak{B}^{(3)}$ with

$$A^{(3)} = p \begin{pmatrix} 1, & 0 \\ 0, & 5.33 \end{pmatrix} + p^{-1} \begin{pmatrix} 4, & -4 \\ -4, & 5.33 \end{pmatrix}$$

$$\equiv p \begin{pmatrix} L_2^{(3)}, & 0 \\ 0, & L_4^{(3)} \end{pmatrix} + p^{-1} \begin{pmatrix} D_2^{(3)}, & -D_2^{(3)} \\ -D_2^{(3)}, & D_2^{(3)} + D_4^{(3)} \end{pmatrix};$$

the point $(0, -1.33)$ labeled P_4 corresponds to the (Cauer) equivalent network $\mathfrak{B}^{(4)}$ with

$$A^{(4)} = p\begin{pmatrix} 4, & -4 \\ -4, & 5.33 \end{pmatrix} + p^{-1}\begin{pmatrix} 1, & 0 \\ 0, & 5.33 \end{pmatrix}$$

$$= p\begin{pmatrix} L_2^{(4)}, & -L_2^{(4)} \\ -L_2^{(4)}, & L_2^{(4)} + L_4^{(4)} \end{pmatrix} + p^{-1}\begin{pmatrix} D_2^{(4)}, & 0 \\ 0, & D_4^{(4)} \end{pmatrix}.$$

the point $(-.33, 1.33)$ labeled P_5 corresponds to the network $\mathfrak{B}^{(5)}$ with

$$A^{(5)} = p\begin{pmatrix} 6.33, & -5.33 \\ -5.33, & 5.33 \end{pmatrix} + p^{-1}\begin{pmatrix} 1.33, & -1.33 \\ -1.33, & 5.33 \end{pmatrix}$$

$$= p\begin{pmatrix} L_2^{(5)} + L_4^{(5)}, & L_4^{(5)} \\ L_4^{(5)} & L_4^{(5)} \end{pmatrix} + p^{-1}\begin{pmatrix} D_2^{(5)}, & -D_2^{(5)} \\ -D_2^{(5)}, & D_2^{(5)} + D_4^{(5)} \end{pmatrix}.$$

To every point (x, y) of W_{12}, a network corresponding to A is realizable without ideal transformers. The point $(.6, 1)$, marked P_6 has a network $\mathfrak{B}^{(6)}$ corresponding to it with

$$A^{(6)} = p\begin{pmatrix} 4.72, & -1.2 \\ -1.2, & 3 \end{pmatrix} + p^{-1}\begin{pmatrix} 2.08, & .18 \\ .18, & 3 \end{pmatrix}$$

$$= p\begin{pmatrix} L_1^{(6)}, & M^{(6)} \\ M^{(6)}, & L_2^{(6)} \end{pmatrix} + p^{-1}\begin{pmatrix} D_1^{(6)} + D_2^{(6)}, & D_2^{(6)} \\ D_2^{(6)}, & D_2^{(6)} + D_3^{(6)} \end{pmatrix}.$$

Example 4. As a further illustration of the theory, consider the following potentially equivalent passive 2-terminal pair networks as indicated below:

The matrix of the network (a) is

$$D = \begin{pmatrix} D_1 + D_3, & D_3 \\ D_3, & D_2 + D_3 \end{pmatrix}$$

where each element in D is positive and D is positive definite. Adjoin a third row and column as indicated:

$$B = \begin{pmatrix} D_1 + D_3, & D_3, & 0 \\ D_3, & D_2 + D_3, & 0 \\ 0, & 0, & 1 \end{pmatrix}.$$

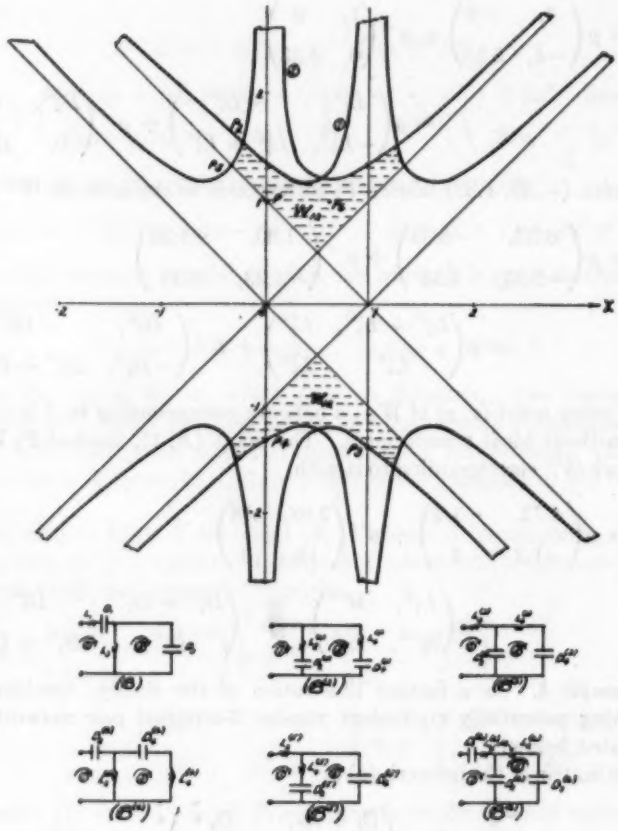

FIG. 3

Then if

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & y & z \end{pmatrix},$$

the networks of matrices

$$A = T^T BT = \begin{pmatrix} D_1 + D_3 + x^2, & D_3 + xy, & xz \\ D_3 + xy, & D_2 + D_3 + y^2, & yz \\ xz, & yz, & z^2 \end{pmatrix}$$

$$= \begin{pmatrix} D_2', & 0, & D_2' \\ 0, & D_1' + D_3' + D_3', & -D_1' \\ D_2', & -D_1', & D_1' \end{pmatrix},$$

are equivalent to (*a*).

If A is an R-matrix, a network corresponding to A does not require an ideal transformer for realizability, and

$$\begin{cases} (D_1 + D_3 + x^2) - |D_3 + xy| - |xz| \geqq 0, \\ -|D_3 + xy| + (D_2 + D_3 + y^2) - |yz| \geqq 0, \\ -|xz| - |yz| + z^2 \geqq 0. \end{cases}$$

The surfaces $\chi_i^{(k)} = 0$ corresponding to these inequalities bound a region W in 3-space, within which all points (x, y, z) make A an R-matrix.

We shall interest ourselves with the special case where $a_{12} = 0$. This means that we are interested only in that portion of the region W common to the cylinder $D_3 + xy = 0$.

We have represented this region W in figure 4, where it is understood that $y = -D_3/x$.

In figure 4, the region W is clearly indicated for the case when

$$D_1 = 1, \qquad D_2 = .5, \qquad D_3 = 2.$$

The curves bounding this region are given by

$$\begin{cases} xy = -2, \\ |xz| \leqq 3 + x^2, \\ |yz| \leqq 2.5 + y^2, \\ |xz| + |yz| \leqq z^2. \end{cases}$$

In order that network (*b*) be equivalent to (*a*), we must have

$$\begin{cases} D_3 + xy = 0, \\ D_1 + D_3 + x^2 = xz, \\ D_2 + D_3 + y^2 = -yz, \end{cases} \quad \text{or} \quad \begin{cases} x = \pm\sqrt{\dfrac{D_1 D_3}{D_2}}, \qquad y = \mp\sqrt{\dfrac{D_2 D_3}{D_1}}, \\ z = \pm\dfrac{D_2 D_3 + D_3 D_1 + D_1 D_3}{\sqrt{D_1 D_2 D_3}}. \end{cases}$$

In figure 4, this case occurs when $x = 2$, $y = 3.5$, that is at point P. However, all networks corresponding to A with (x, z) within W are also equivalent to (a) and do not require ideal transformers for realizability.

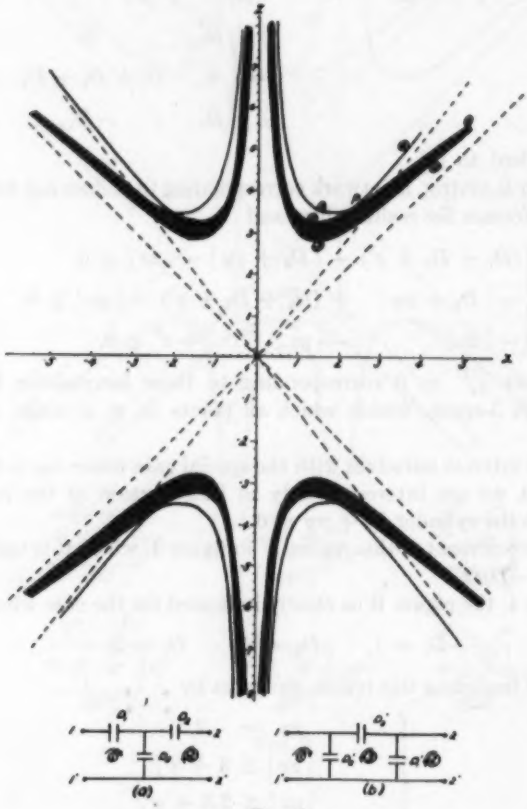

FIG. 4

8. **Conclusions.** Further extensions of the theory and application of R-matrices will be made in another paper. However, the present development should be of some use and should cast light upon the theory of equivalent networks.

Obviously, it is impossible, in a certain sense, to construct models of

the various surfaces $\chi_q^{(k)} = 0$ unless the number of parameters to be considered is less than or equal to 3.

However, such models can be constructed quite easily for the cases where two or three parameters are allowed to vary, the others being fixed.

Moreover, in dealing with equivalent electrical networks, we are most frequently concerned with the equivalence of 1-terminal or 2-terminal pair networks, which dictates that T be 1-affine or 2-affine, thus reducing the number of parameters in T.

Let $T = T_1, \cdots, T_q$ where T_j contains but one parameter. A region W_1 of realizability for the parameter in T_1 may be constructed for $A_1 = T_1^\tau B T_1$; similarly within W_1, a new region W_2 may be generated by T_2 for $A_2 = T_2^\tau A_1 T_2$, etc. This method does not lead to the most general region of realizability for T, but may frequently be useful.

This paper then gives a method for selecting ranges for the parameters of T such that the networks of matrices $A = T^\tau B T$ will not require ideal transformers for realizability.

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ON PRODUCT INTEGRATION

By Garrett Birkhoff

I. Introduction

1. **Homogeneous linear differential equations.** Let us recall how the theory of systems of homogeneous linear differential equations looked[1] in 1840.

By regarding derivatives as new variables, we can replace any such system by one of the first order, having the general form

$$dx_i/dt = \sum_{k=1}^{n} x_k(t)a_{ki}(t) \qquad [i = 1, \cdots, n] \tag{1}$$

Moreover there is one and only one solution of (1) which satisfies a given set of boundary conditions $x_i(0) = a_i [i = 1, \cdots, n]$. Further, if for each fixed i, we denote by $b_{ki}(t)$ the solution of (1) which satisfies the set of boundary conditions[2] $b_{ki}(0) = \delta_k^i$ $[k = 1, \cdots, n]$, then

$$x_i(t) = \sum_{k=1}^{n} a_k b_{ki}(t) \tag{2}$$

satisfies (1) and also $x_i(0) = a_i$. That is, (2) gives the *general* solution of (1).

2. **Relation to the algebra of matrices.** These facts (stated in various forms) appear in many texts which ignore the notion of product integration. To bridge the gap, two observations are needed.

First, the $a_{ki}(t)$ and $b_{ki}(t)$ define matrix functions[3] $A(t)$ and $B(t)$ of the

[1] Cf. Cauchy's "Leçons de calcul différentiel et intégral", pub. by Moigno, Paris (1844), Lesson 36.

[2] Of course δ_k^i is the Kronecker delta-function: one if $i = k$ and zero otherwise —$\| \delta_k^i \|$ is the matrix unity.

[3] In the sense of Cayley (1857). We shall adopt the usual notation $X = \| x_{ij} \|$, $Y = \| y_{ij} \|$, $\cdots$ for matrices. The "null-matrix" $\| 0 \|$ will be denoted by 0; the "unit-matrix" $\| \delta_j^i \|$ by I; the "sum" $\| x_{ij} + y_{ij} \|$ of two matrices X and Y by $X + Y$; the "product" $\left\| \sum_{j=1}^{n} x_{ij} y_{jk} \right\|$ of X and Y, by $X \circ Y$; the product $\| x_{ik}\lambda \|$ of X by a "scalar" λ, by $X\lambda$. For the laws of matrix algebra, the reader is referred to Wedderburn's "Lectures on matrices", pp. 4–5. They are in the

variable t (which may be thought of as representing the lapse of time). And second, to within infinitesimals of higher order (provided the $a_{ji}(t)$ are continuous),

$$b_{ki}(t + \Delta t) \simeq b_{ki}(t) + \sum_{j=1}^{n} b_{kj}(t)a_{ji}(t)\Delta t \tag{3}$$

which, in the notation of matrix algebra, becomes

$$B(t + \Delta t) \simeq B(t) \circ [I + A(t)\Delta t)] \tag{4}$$

and resembles closely the relation

$$x(t + \Delta t) \simeq x(t) + a(t)\Delta t \tag{5}$$

which determines the solution $x(t)$ of the differential equation $dx/dt = a(t)$. In the latter case (of so-called "quadrature"), the lapse of time Δt changes the integral by the *addition* of a small quantity; in the case of homogeneous linear differential equations, the lapse changes the integral through *multiplication* by a matrix differing slightly from the unit matrix.

These reflections are what must have occurred to V. Volterra[4] (1887). They suggest calling $B(t)$ the "product[5] integral" of $A(t)$, and inversely, calling $A(t)$ the "product derivative" of $B(t)$. Thus they suggest developing an *infinitesimal calculus of matrices* as a self-contained theory, in which the $x_k(t)$ do not appear at all.

3. **Sketch of the infinitesimal calculus of matrices.** One can easily define the *derivative* (properly, right-derivative)[6] of a general matrix function $B(t)$, so that it is $A(t)$ in the case we have discussed. Namely, one can define the derivative to be the limit

$$A(t) = \operatorname{Lim}_{\Delta t \to 0} [B^{-1}(t) \circ B(t + \Delta t) - I]/\Delta t \tag{6}$$

---

main the same as those of ordinary algebra, except that (a) in general $X \circ Y \neq Y \circ X$ (non-commutativity), (b) some "singular" matrices X have no inverse X^{-1} with $X \circ X^{-1} = X^{-1} \circ X = I$.

[4] "Sulle equazioni differenziali lineari", Rend. dei Lincei 3 (1887), 391–6. Also "Sui fondamenti della teoria delle equazioni diff. lineari", Mem. Soc. Ital. Sci. III 8 (1887), p. 6.

[5] Since matrix multiplication is non-commutative, it is proper to speak of product integration as "non-commutative" integration.

[6] By inverting the order of multiplication, one can define a symmetrical left-derivative.

Can one inversely, given $A(t)$, find an "integral" of which it is the derivative in the sense of (6)? One can[12] by defining

$$\int_a^b A(t)\,dt = \mathrm{Lim}_{|\pi|\to 0} \prod_{k=1}^{r(\pi)} [I + A(t_k)\,|\,\Delta_k\,|] \tag{7}$$

where π denotes a partition of $[a, b]$ into segments $\Delta_1, \cdots, \Delta_{r(\pi)}$ of lengths $|\Delta_k| \leqq |\pi|$, $t_k \in \Delta_k$, and $\prod_{k=1}^{r} X_k$ is an abbreviation for $X_1 \circ \cdots \circ X_r$.

It is easy to see that the limit in question exists provided $A(t)$ is (uniformly[7]) continuous. The relation

$$(X + \Delta X)\circ(Y + \Delta Y) = X\circ Y + (X\circ\Delta Y + Y\circ\Delta X + \Delta X\circ\Delta Y) \tag{8}$$

shows that a small change in each of a set of factors produces a small change in their product. And the relation

$$(I + \Delta X)\circ(I + \Delta Y) = (I + \Delta X + \Delta Y) + \Delta X\Delta Y \tag{9}$$

(together with the small oscillation of $A(t)$ on each small Δ_k) shows that $[I + A(t_k)\,|\,\Delta_k\,|\,]$ is replaced when Δ_k is subdivided, by something differing little from it.

We shall show later (§§8–13) under more general assumptions, that these definitions of integral and derivative *are* reciprocal. Moreover, they *do* yield the general integral (2) of (1) in §1, and have a number of basic formal properties. Finally, even the Lebesgue integral can be extended (provided the $a_{ik}(t)$ are bounded and measurable) so as to apply to matrices. This has been shown by L. Schlesinger,[8] who, however, neglected to prove the reciprocity of L-integrability and differentiability p.p. This reciprocity will be established below (§25).

4. The infinitesimal calculus of transformations. The above sketch outlines what is now known about general methods of product integration; there exists a coherent infinitesimal calculus of matrices. We shall now suggest how Lie's notion[9] of an "infinitesimal transformation" fits into a more general conception of an infinitesimal calculus, with a more general product integral. In this calculus, matrices are replaced by

[7] Continuity on a closed segment implies uniform continuity on that segment. Cf. footnote (27).

[8] "Neue Grundlagen für eine Infinitesimalkalkul für Matrizen", Math. Zeits. 33 (1931), 33–61.

[9] Cf. S. Lie, "Transformationsgruppen", Leipzig, 1888, p. 45.

displacements—which unfortunately have fewer algebraic properties than matrices.

To get an idea of it, imagine a liquid circulating in a glass jar. Let $B(t)$ denote the "displacement" undergone by the liquid during the time interval $[0, t]$—i.e., the transformation carrying each point occupied by a molecule of the liquid at the time $t = 0$, into the point occupied by the same molecule at the time t. Then the displacement undergone by the liquid during *any* time-interval $[t, u]$, is the transformation[10] $B^{-1}(t) \cdot B(u)$. Further, the *velocity* of flow of the liquid at any instant t, is the *vector-field*

$$V(t) = \mathrm{Lim}_{\Delta t \to 0} \frac{1}{\Delta t}[B^{-1}(t) \circ B(t + \Delta t)] \tag{10}$$

toward which the small displacements $B^{-1}(t) \circ B(t + \Delta t)$, when multiplied by the factor $\frac{1}{\Delta t}$, converge as Δt approaches zero.[11]

In the language of Lie, $V(t)$ is the *infinitesimal transformation* associated at the instant t with the one-parameter family (which need not be a group!) of transformations $B(t)$—and a comparison of formulas (6) and (10) reveals that it specializes to Volterra's notion of the derivative of a matrix when the $B(t)$ are linear and homogeneous. (Cf. §7, end.)

This suggests trying to generalize the theory of product integration so as to apply to *non-linear* as well as linear transformations, and to show that the generalized theory yields existence and uniqueness theorems on the solutions of systems of *non-linear* (as well as of linear homogeneous) ordinary differential equations. These objectives are attained in the present paper; integrals of both the Riemann and the Lebesgue type are set up.

5. **Extension to infinite dimensions.** Furthermore, the theory is presented in such a form that it applies as it stands to Banach spaces. This seems to the author to have great potential importance: it means that it applies to phase-spaces of infinitely many degrees of freedom.

[10] We denote by $B^{-1}(t)$ the "inverse" of the transformation $B(t)$, and by $X \circ Y$ the "product" of the transformations X and Y.

[11] We have purposely avoided giving a precise meaning of what is meant by convergence, or by multiplying a displacement by a scalar factor. The latter is suggested by the picture of a displacement, obtained when one draws a short arrow from each point to the point into which it is displaced. Cf. §7.

Actually, an application to the theory of transition probabilities is made in the present paper.

Moreover, the theory becomes simpler even in the finite-dimensional case when coordinates (which are extrinsic anyway) are eliminated—and such expressions as $\sum_{k=1}^{n} x_k a_{ki}$ replaced by such expressions as xA. In the case of most Banach spaces, coordinates are difficult to handle besides.

II. The Riemann Integral

6. **Definition.** Many processes of integration have their genesis in the *inversion* of (10). Namely, given $V(t)$, each division π of the segment $[a, b]$ over which $V(t)$ is to be integrated yields an *approximation*

$$J_\pi(V) = B(A) \circ \prod_{k=1}^{n} [V(t_k) \cdot |\Delta_k|] \tag{11}$$

to $B(b)$. Presumably the approximation gets closer as π becomes finer (in symbols, as $\pi \to 0$), and so we *define*

$$\int_a^b V(t)\, dt = \operatorname{Lim}_{\pi \to 0} J_\pi(V) \tag{12}$$

as the *Riemann*[12] product-integral of $V(t)$ over $[a, b]$.

We shall now discuss the precise interpretation of the phrase "as π approaches 0".

The usual interpretation[13] is the following. Each π is associated with the positive number $|\pi| = \sup |\Delta_k|$, and $J_\pi(V)$ is said to ap-

[12] For short, R-integral. Historical note: This so-called "polygon method" was used by Euler ("Calcul intégral", St. Petersburg, 1769) both for quadratures (pp. 200–29) and for integrating systems of ordinary differential equations (pp. 493–510). Cauchy gave a fairly rigorous justification of it (cf. Moigno's "Leçons," Vol. II, pp. 1–5 and 385–96). As perfected by Lipschitz ("Lehrbuch der Analysis", Bonn, 1877, Vol. II, p. 504), it still provides the standard existence theorem on integrals of ordinary differential equations. Goursat's "Cours d'Analyse", Paris (1908), Vol. II, p. 376, or Chap. I of G. D. Birkhoff's "Dynamical systems", New York, 1927.

Euler and Cauchy required t_k to be the left extremity of Δ_k. Riemann ("Über die Darstellbarkeit einer Funktion durch trig. Reihen" (1854), Werke, pp. 213–53) let the t_k be any point of Δ_k, treated the case of quadrature in full, and gave his name to the integral. Volterra (op. cit.) used precisely this construction for integrating matrices.

[13] Cf. Euler, op. cit., p. 203. It is also that of Riemann.

proach B as $\pi \to 0$ if and only if, to each neighborhood $U(B)$ of B corresponds a positive number δ so small that $|\pi| < \delta$ implies that $J_\pi(V)$ is in $U(B)$.

But we shall adopt a different interpretation, due to E. H. Moore.[14] We shall say that $J_\pi(V) \to B$ as $\pi \to 0$ if and only if, given $U(B)$, $\pi(U)$ exists so fine that $J_\pi(V)$ is in $U(B)$ for all *subpartitions*[15] π of $\pi(U)$.

7. **The algebra of displacements.** What are the most general conditions under which (10) and the inverse construction of §6 have a sense? They certainly do if the values of the integrand (1) constitute a linear space (2) constitute a "groupoid"[16] with respect to multiplication (3) possess a topology which is not too pathological. If this is the case, then[17] no $V(t)$ has more than one integral.

We shall now see that the "displacements" of *any* linear space L have at least the necessary algebraic properties (1)–(2).

The linearization of the displacements of L is obtained as follows. Picture each transformation X of L into itself, as the set of arrows (or "vectors") joining every point $p \in L$ with its "image" $X(p)$. Then the "sum" $X = Y + Z$ of two displacements is determined by the identity

$$X(p) - p = [Y(p) - p] + [Z(p) - p] \tag{13}$$

and the "scalar product" $X\lambda$ of X by any number λ, from the identity

$$X\lambda(p) - p = [X(p) - p]\lambda \tag{14}$$

That is, the vector algebra of displacements is simply the vector algebra of the imaginary vectors we have drawn.

The "product" of two displacements of L is simply their transformation product: i.e. $X = YZ$ means that $X(p) = Z(Y(p))$ for all p. It is clear that $X(YZ) = (XY)Z$, and that the displacement 0 of the vector space of the preceding paragraph, satisfies $OX = XO = X$ for all X.

[14] "Definition of limit in general integral analysis", Proc. Nat. Acad. Sci., vol. 1 (1915), p. 628.

[15] By a "subpartition" of $\pi(U)$ is meant one which can be obtained by subdividing the pieces into which π divides $[a, b]$. This notion is basic in Cauchy's theory (cf. Moigno, p. 4).

[16] By a "groupoid", we mean a system in which an associative multiplication is defined, and which possesses an "identity" I satisfying $IX = XI = X$ for all X.

[17] This is ensured by the fact that every two partitions have a common subpartition (namely, their product)—whence if two integrals existed, their neighborhoods would all overlap, in contradiction to Hausdorff's fourth axiom. (Trennungsaxiom).

Thus 0 is a unity I for multiplication, identifying the notation of (6)–(7) with that of (10)–(11).

The identities of the last two paragraphs yield most of the "algebra of displacements," and we shall call any abstract system satisfying them, a "linearized groupoid."

8. **Digression: a realization theorem.** The algebra of displacements evidently resembles matrix algebra—with the preliminary qualification that the *displacement* sum of two linear operators X and Y, is *not* their sum under the algebra of linear operators: in terms of the latter algebra, it is $I + (X - I) + (Y - I)$.

As we have developed it, the *distributive* laws of matrix algebra are also absent; they do not even have any analogs. But actually, we *do* have a sort of *right*-distributivity; it is the left-distributive law which totally disappears when we deal with non-linear operators.

The modified right-distributive law is

$$(D) \qquad X(A + B) - X = (XA - X) + (XB - X)$$

[Proof: Note first that $p(Y + Z) = pY + pZ - p$ and that $p(Y - Z) = p + pY - pZ$. Using these, direct computation shows that both sides of (D) carry any p into the same element $p + pXA + pXB - 2pX$.]

We also have the right-linearity law

$$(D') \qquad X(A\lambda) - X = [XA - X]\lambda$$

(both sides carry any p into $p + (pX\alpha - pX)\lambda$).

Theorem 1: Any linearized groupoid G satisfying (D)–(D') can be realized by displacements.

Proof: We shall use a classic construction due to Cayley,[18] and associate each element A in G with the transformation $T_A: X \to XA$ of G. Then T_0 is the identity; distinct elements induce distinct transformations since $T_X = T_Y$ implies $OX = OY$; the associative law implies $T_{AB} = T_A T_B$; (D) implies $T_{A+B} = T_A + T_B$; and (D') implies $T_{\lambda A} = (T_A)\lambda$.

The significance of Theorem 1 is, that any algebraic identity on displacements can be deduced from the special identities we have listed.

[18] "On the theory of groups", Phil. Mag. 7 (1854), 40-7. Cayley used it to obtain the analog to Theorem 1 for finite groups; Poincaré (1884) used it to obtain the analog for linear associative algebras. It is the cornerstone of modern "representation theory".

It shows that we have a complete set of identities for the algebra of displacements.

We postponed the discussion of $(D) - (D')$ until the present digression, as they are nowhere used in our theory of integration.

9. **Key to the theory.** The theory of integration developed below hinges on the following principle: the $J_\pi(X)$ of §6 converge to a $J(X)$ provided (1) the differences $J_\pi(X) - J_{\pi'}(X)$ are small for all subpartitions π' of suitable partitions π, and (2) the space in which the $J_\pi(X)$ lie is *complete*.

To see this when (as below) the space is *metric*,[19] choose a sequence of partitions π_n such that $| J_{\pi_n}(X) - J_{\pi_n'}(X) |$ is less than $1/n$ for all $\pi_n' \leqq \pi_n$. Then since the product[20] $\pi_{m,n}$ of π_m and π_n is a subpartition of both π_m and π_n, by the triangle inequality $| J_{\pi_m}(X) - J_{\pi_n}(X) | \leqq 1/m + 1/n$. Hence by the assumption of completeness, the $J_{\pi_n}(X)$ converge to some $J(X)$. Finally, if π_n^* is any other such sequence, then $| J_{\pi_n}(X) - J_{\pi_n^*}(X) | \leqq 1/n + 1/n$, and so $J_{\pi_n^*}(X) \to J(X)$. We conclude that $J(X)$ is *unique*, completing the proof.

10. **Remaining assumptions.** Thus if we assume that the space in which the $J_\pi(X)$ lie is complete, we need only show that $J_\pi(X) - J_{\pi'}(X)$ is small for all subpartitions π' of suitable partitions π.

Accordingly, suppose π divides the domain $[a, b]$ of integration into segments $\Delta_1, \cdots, \Delta_n$, and that π' subdivides each Δ_k into subintervals Δ_k^n. Then any $J_{\pi'}(X)$ will assume the form $\prod_{k=1}^{n} \left(\prod X(t_n^k) \cdot | \Delta_k^n | \right)$, and any $J_\pi(X)$, the form $\prod_{k=1}^{n} X(t_k) \cdot | \Delta_k |$.

Hence if we assume also that group multiplication is *locally uniformly continuous*,[21] then we need only show that the sum (over k) of the differences $\prod_n X(t_k^n) \cdot | \Delta_k^n | - X(t_k) \cdot | \Delta_k |$ is small.

[19] The notions of a metric space and a complete metric space are of course due to M. Frechet, "Sur quelques points dûe Calcul Fonctionnel", Rend. di Palermo 22 (1906), 1–74. The notion of completeness really goes back to Cauchy. For definitions, cf. F. Hausdorff "Mengenlehre" (1927), p. 94, p. 103.

[20] In the usual sense: the partition dividing the domain of integration into the intersections of the subdivisions yielded by π_m, with those yielded by π_n.

[21] More precisely, that group translations $X \to AXB$ increase differences $X - Y$ in bounded ratio, for all A, B which can occur in the products considered (i.e., for A, B bounded).

But these differences can be split into two parts,

$$\prod_a X(t_k^a)\cdot|\Delta_k^a| - \sum_a X(t_k^a)\cdot|\Delta_k^a| \quad \text{and} \quad \sum_a X(t_k^a)\cdot|\Delta_k^a| - X(t_k)\cdot|\Delta_k|.$$

The first of these is small provided *sums are good approximations to products* over short stretches. And since $\sum_a X(t_k^a)\cdot|\Delta_k^a|$ lies in the convex hull[22] of the set of the $X(t_k)\cdot|\Delta_k|$, the second is small provided the topology is *convex* (as is always the case below[22]), and $X(t)$ has a small oscillation on each Δ_k—i.e., is (uniformly) *continuous*.

11. **Metric algebra of displacements.** Now let B be any Banach space.[23] The displacements of B have enough metric properties to permit the argument of §§9–10 to go through.

Firstly, any displacement X has a (finite or infinite) "displacement modulus" $|X| = \sup_{p\in B}|pX - p|$, and a (finite or infinite) "Lipschitz modulus"

$$(15) \qquad \|X\| = \sup_{p\neq q}|(pX - qX) - (p - q)|/|p - q|$$

We shall call $|X|$ and $\|X\|$ the "D-modulus" resp. "L-modulus" of X, for short.

Relative to the vector calculus of displacements, and to D-modulus as norm, the displacements with finite D-modulus form a new Banach space Ω_B; we shall omit the proof. Furthermore, the displacements which in addition[24] have a finite L-modulus, form a linear subspace Ω_B' of Ω_B, which is not however closed (and therefore not complete). Thus Ω_B' is a metric linear space relative to $|X|$ as norm; it is almost[25] a metric linear space relative to $\|X\|$ as norm. But it is *not* a Banach space: it is not complete.

[22] By the "convex hull" of any set, is meant the set of centers of gravity of distributions of mass on the set. A topological linear space is called "convex", if and only if its convex neighborhoods define its topology. Any metric linear space is convex, since the spheres $|x - a| < \rho$ are convex.

[23] I.e., complete metric linear space. A linear space is called metric if and only if its elements x have a norm $|x|$ which satisfies (1) $|x| = 0$ if and only if $x = 0$, (2) $|\lambda x| = |\lambda|\cdot|x|$ and (3) $|x + y| \leq |x| + |y|$. In any such space, the distance $|x - y|$ is metric. A Banach space is logically just a crudely defined Euclidean space.

[24] Displacements of a bounded region of a metric linear space having a finite L-modulus, necessarily have a finite D-modulus.

[25] However $\|X\| = 0$ does not imply $X = 0$; it merely implies that X is a translation, i.e., that $pX = p + OX$.

However, it *does* possess a kind of completeness: the convex subsets Ω_B^M of Ω_B' consisting of all elements X with $\| X \| \leqq M$, are *closed*[36] (and hence metrically complete). For if $| X_n - X | \to 0$ as $n \to \infty$, then

$$| pX - qX - (p - q) | = \operatorname{Lim}_{n\to\infty} | pX_n - qX_n - (p - q) |$$
$$\leqq M \cdot | p - q |$$

identically, and so $\| X \| \leqq M$. Thus $\| X \|$ is lower semicontinuous on Ω_B.

Again, the product XY of any two displacements in Ω_B' is again in Ω_B'; in fact,

M1: $\quad | XY | \leqq | X | + | Y |$ and

$$\| XY \| \leqq \| X \| + \| Y \| + \| X \| \cdot \| Y \|$$

[Proof: $| pXY - p | \leqq | pXY - pX | + | pX - p |$, while

$$| pXY - qXY - (p - q) | \leqq | pXY - qXY - (pX - qX) |$$
$$+ | pX - qX - (p - q) | \leqq \| Y \| \cdot | pX - qX | + \| X \| \cdot | p - q |$$

which, since $| pX - qX | \leqq | p - q | + | pX - qX - (p - q) | \leqq (1 + \| X \|) \cdot | p - q |$, yields the second inequality of M1.]

Finally, the conditions of §10 are fulfilled in each Ω_B^M. They are contained (cf. 12) in the inequalities

M2: $$| XY - XZ | \leqq | Y - Z |$$

M3: $$| (YX - ZX) - (Y - Z) | \leqq \| X \| \cdot | Y - Z |$$

Proof of M2: Set $q = pX$; then

$$| XY - XZ | \leqq \sup_q | qY - qZ | = | Y - Z |$$

Proof of M3: Set $r = pY, r' = pZ$. Then

$$| (YX - ZX) - (Y - Z) | = \sup | (rX - r'X) - (r - r') |$$
$$\leqq \| X \| \cdot \sup | r - r' | \leqq \| X \| \cdot | Y - Z |$$

which completes the proof.

12. **Expansions by induction.** In the present section, we shall adapt relations M1–M3 for future use.

[36] Thus Ω_B' is an F_σ, in the sense of Hausdorff, "Mengenlehre", 2d ed., p. 136. For lower semi-continuity, cf. ibid., p. 248.

The reader can see that M1 implies when expanded,

$$\left|\prod_{k=1}^{n} X_k\right| \leqq \sum_{k=1}^{n} |X_k| \tag{16}$$

$$\left\|\prod_{k=1}^{n} X_k\right\| \leqq \exp\left\{\sum_{k=1}^{n} \|X_k\| - 1\right\} \tag{16'}$$

Moreover the uniform continuity required in §10 follows from M2–M3, when these are combined to yield

$$|AXB - AYB| \leqq |XB - YB| \leqq (1 + |B|)\cdot|X - Y| \tag{17}$$

We can easily expand (17) by induction, and get

$$\left|\prod_{k=1}^{n} X_k - \prod_{k=1}^{n} Y_k\right| = \sum_{k=1}^{n} |A_k X_k B_k - A_k Y_k B_k| \leqq \exp\left\{\sum_{k=1}^{n} \|X_k\|\right\}\cdot\left(\sum_{n=1}^{n} |X_k - Y_k|\right) \tag{17*}$$

writing A_k for $\prod_{k=1}^{k-1} Y_k$, B_k for $\prod_{k+1}^{n} X_k$, and using the triangle inequality, (17), and (16′).

This shows why a Lipschitz condition is needed; in Ω'_n, multiplication is not even continuous.

Finally, the last assumption of §10 can be had from M3 by setting $Z = 0$ and interchanging X with Y. This yields

$$|(XY) - (X + Y)| \leqq |X|\cdot\|Y\| \tag{18}$$

Moreover using the triangle inequality, we get

$$\left|\prod_{k=1}^{n} X_k - \sum_{k=1}^{n} X_k\right| \leqq \sum_{k=1}^{n} \left|\prod_{k=1}^{n} X_k - \left[X_k + \left(\prod_{k+1}^{n} X_k\right)\right]\right| = \sum_{k=1}^{n} |X_k|\cdot\left[\exp\left\{\sum_{k=1}^{n} \|X_k\|\right\} - 1\right] \tag{18*}$$

by (18) and (16′).

13. **Digression: refinements.** We shall now deepen Theorem 1, by showing

Theorem 2: In the realization of Theorem 1, M2-M3 imply that $|T_A| = |A|$ and $\|T_A\| \leqq \|A\|$.

Proof: By M2, setting $Z = 0$, and $Y = A$, we have $|XA - X| \leqq |A|$,

whence $|T_A| \leqq |A|$. But $|OA - O| = |A|$, and so $|T_A| = |A|$. Relation M3 asserts $||T_A|| \leqq ||A||$—by the very definition of L-modulus and T_A.

Theorem 3: Every element X with L-modulus less than unity, has a (unique) inverse X^{-1}.

Proof: Let X be given; set $Y_1 = -X$ and (using induction) $Y_{k+1} = Y_k - Y_kX$. By induction, we get

$$\begin{aligned} |Y_{k+1}X| &= |(Y_{k+1}X - Y_kX) - (-Y_kX)| \\ &= |(Y_{k+1}X - Y_kX) - (Y_{k+1} - Y_k)| \leqq ||X|| \cdot |Y_kX| \end{aligned}$$

Hence if $||X|| < 1$, the sequence $\{Y_k\}$ converges with the rapidity that $||X||^k$ converges to 0, to a limit Y. By M3, we have

$$YX = (\operatorname{Lim} Y_k)X = \operatorname{Lim}(Y_kX) = 0,$$

and so X has a left-inverse Y. But by M3 again,

$$\begin{aligned} |XY - O| &= |(XO - OX) + (XY - O)| \\ &= |(XYX - OX) - (XY - O)| \leqq ||X|| \cdot |XY - O| \end{aligned}$$

and so $(1 - ||X||)(|XY - O|) \leqq 0$, implying $|XY - O| = 0$ and $XY = 0$.

Remark: Using Theorem 2, we can easily show that $|X^{-1}| = |X|$ and $||X^{-1}|| \leqq ||X||/(1 - ||X||)$ in Theorem 3.

The above results depend only on M1–M3 and the completeness condition; in fact, the first part of M1 is a corollary of M2. We now come to two inequalities of less interest, which the author was unable to deduce from M1–M3.

The first is a condition analogous to M2,

M2′: $||XY - XZ|| = ||Y - Z||/(1 - ||X||)$, provided $||X|| < 1$.

[Proof: $||XY - XZ|| = \sup\{|(pXY - pXZ) - (qXY - qXZ)|/|p - q|\}$. Now set $pX = r$, $qX = r'$; then $|r - r'| \cdot (1 - ||X||) \leqq |p - q|$, and so $||XY - XZ|| \leqq \sup|[r(Y - Z) - r'(Y - Z)] - (r - r')|/|r - r'| \cdot (1 - ||X||)$ from which M2′ is obvious.

The second is a symmetric dual to M3,

M3′: $$|(XY - XZ) - Y - Z)| = |X| \cdot ||Y - Z||$$

[Proof: Note that $|V - W| = \sup \cdot |pV - pW|$. Hence if we set $U = Y - Z$ and $q - pX$, we have

$$|(XY - XZ) - (Y - Z)| = \sup|(qU - q) - (pU - p)|$$
$$= ||U|| \cdot |q - p| = ||U|| \cdot |X|$$

14. Integrability of continuous functions. Fitting together the ideas of §§9–10 and the inequalities of §12, one can easily prove the basic

Theorem 4: If $X(t)$ varies continuously with t in the metric $|X - Y|$, and if $||X(t)||$ is bounded, then $X(t)$ is R-integrable on any finite domain $[a, b]$.

Proof: As in real function theory, continuity implies uniform continuity;[27] hence if $|\pi|$ is sufficiently small, the oscillation of $X(t)$ on every Δ_k can be bounded by any $\epsilon > 0$. It follows by the convexity of the norm $|X|$, that $\left|\sum_\alpha X(t_k^\alpha) \cdot |\Delta_k^\alpha| - X(t_k) \cdot |\Delta_k|\right| < \epsilon \cdot |\Delta_k|$.

Again, letting $M = \sup||X(t)||$ and $M^* = \sup|X(t)|$ (the finiteness of M^* follows by uniform continuity), we get

$$\left|\prod_\alpha X(t_k^\alpha) \cdot |\Delta_k^\alpha| - \sum_\alpha X(t_k^\alpha) \cdot |\Delta_k^\alpha|\right| \leq M^* \cdot |\Delta_k| \cdot (e^{M|\pi|} - 1)$$

Combining these two inequalities, we have

$$\sum_{k=1}^n \left|\left(\prod_\alpha X(t_k^\alpha) \cdot |\Delta_k^\alpha|\right) - X(t_k) \cdot |\Delta_k|\right| \leq \epsilon \cdot |b - a| + M^*(e^{M|\pi|} - 1)$$

Now using (17*), we get finally

$$|J_\pi(X) - J_{\pi'}(X)| \leq e^{M|b-a|}[\epsilon \cdot |b - a| + M^*(e^{M|\pi|} - 1)]$$

which tends to zero with $|\pi|$, completing the proof.

Remark 1: We have not only shown that $\mathrm{Lim}_{\pi \to 0} J_\pi(X)$ exists, but even that $\mathrm{Lim}_{|\pi| \to 0} J_\pi(X)$ does—the limit exists in the sense of Riemann, and not merely in that of E. H. Moore.

Remark 2: Actually, Theorem 4 can be proved in this way for all functions "of class (R)"—such that $||X(t)||$ is bounded and the points where Osc $X(t)$ exceeds $\epsilon > 0$ can be enclosed in intervals $\Delta_1, \cdots, \Delta_{n(\epsilon)}$, the sum of whose lengths is arbitrarily small.

[27] If $X(t)$ were not uniformly continuous, one could find a sequence of subintervals $[u_n, v_n]$ converging to a limit u, such that the oscillation of $X(t)$ on any $[u_n, v_n]$ exceeded $\epsilon > 0$. Hence $X(t)$ would be discontinuous at u, contrary to hypothesis.

15. **Properties of the integral.** We now naturally ask: what general properties does the integral possess—besides making continuous functions with bounded L-modulus integrable?

Theorem 5: $\int_a^b Odt = 0$, irrespective of $[a, b]$.

Proof: $J_\pi(X)$ is identically 0.

Theorem 6: If $a \leqq b \leqq c$, and if $\int_a^b X(t)\,dt$ and $\int_b^c X(t)\,dt$ exist, then $\int_a^c X(t)\,dt$ exists and is their product.

Proof: Partitions π_1, π_2 of $[a, b]$ and $[b, c]$ exist making $J_{\pi_k'}(X)$ arbitrarily near the corresponding integrals, for all subpartitions π_k' of the $\pi_k [k = 1, 2]$. Combining π_1 and π_2, we get a partition π of $[a, c]$ such that $J_{\pi'}(X) = J_{\pi_1'}(X) J_{\pi_2'}(X)$ differs arbitrarily little from the product of the integrals for all subpartitions π' of π. This proves the theorem (actually, we only used the fact that multiplication was continuous!).

Theorem 7: Under the hypotheses of Theorem 4, $\| Y(u) \| = \left\| \int_a^u X(t)\,dt \right\| \leqq \exp \{ | b - a | \cdot \sup \| X(t) \| \}$. Moreover, $Z(u) = \int_{b-a}^{b-a} -X(b - t)\,dt$ exists and is inverse to $Y(u)$.

Proof: By (16′) $\| J_\pi(X) \| \leqq e^{M|b-a|}$ identically, and since each Ω_B^M is closed, we infer $\| Y(u) \| \leqq e^{M|b-a|}$. Again, $Z(u)$ exists since $-X(b - t)$ satisfies the hypotheses of Theorem 4. Finally, to prove $Y(u)Z(u) = 0$, since multiplication is continuous, we need only show that

$$\phi \equiv \left(\prod_{k=1}^{n} X(t_k) \cdot | \Delta_k | \right) \circ \left(\prod_{k=n}^{1} (-X(t_k)) \cdot | \Delta_k | \right)$$

is small when $| \pi |$ is small. But by (17)–(18),

$$| AW(-W)B - AB | \leqq e^{||B||} \cdot \| W \| \cdot | W | \tag{19}$$

Applying this, we get by induction

$$| \phi | \leqq \sum_{k=1}^{n} e^{M|b-a|} \cdot (M \cdot | \pi |) \cdot | X(t_k) | \cdot | \Delta_k | = MM^* \cdot | \pi | \cdot e \tag{20}$$

which tends to zero as $| \pi | \to 0$.

Theorem 8: Under the hypotheses of Theorem 4, $Y(u) = \int_a^u X(t)\,dt$ is everywhere differentiable to $X(u)$.

Proof: We shall prove right—differentiability; left—differentiability will follow by Theorem 7. But by Theorems 6–7,

$$\frac{1}{\Delta u}[Y^{-1}(u)Y(u+\Delta u)] = \frac{1}{\Delta u}\int_u^{u+\Delta u} X(t)\,dt \tag{21}$$

Now approximate to the right-hand side of (27) through $\frac{1}{\Delta u}\prod_{k=1}^{n} X(t_k)\cdot|\Delta_k|$; as in the proof of Theorem 4, this differs from $\frac{1}{\Delta u}\sum_{k=1}^{n} X(t_k)\cdot|\Delta_k|$ by at most $M^*\cdot(e^{M|\Delta u|}-1)$. And the sum differs from $X(u)$ by at most the oscillation of $X(t)$ on $[u, u+\Delta u]$. Passing to the limit, we get

$$\left|\frac{1}{\Delta u}\left[\int_u^{u+\Delta u} X(t)\,dt\right] - X(u)\right| \leq M^*(\cdot e^{M|\Delta u|}-1) + Osc_{\Delta}\, X(t) \tag{22}$$

which tends to zero as $\Delta u \to 0$, q.e.d.

Theorem 9: If $Y(t)$ is differentiable, and if $X(t) = Y'(t)$ satisfies the hypotheses of Theorem 4, then $Y(u) = Y(a)\int_a^u X(t)\,dt \equiv Z(u)$ identically—in words, the indefinite integrals of $X(t)$ are all obtained by multiplying the definite integral by a constant left-factor.

Remark. The analog of this proposition in the theory of real functions is usually proved by appealing to Rolle's Theorem—evidently we must use a very different principle.

Proof: Consider $f(u) = |Y(u) - Z(u)|$, and write $Y(u+\Delta u) = Y(u)\circ(X^*\Delta u)$, $Z(u+\Delta u) = Z(u)\circ(X^{**}\Delta u)$, $X = X(u)$. By Fig. 1,

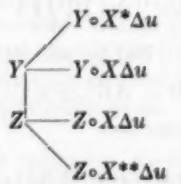

FIG. 1

when u increases to $u + \Delta u$, $f(u)$ increases by at most $|YX^*\Delta u - YX\Delta u| + |ZX^{**}\Delta u - ZX\Delta u| + |(YX\Delta u - ZX\Delta u) - (Y - X)|$. And by M2–M3, this is at most

$$(23) \qquad |X^* - X| \cdot \Delta u + |X^{**} - X| \cdot \Delta u + M\Delta u \cdot |Y - Z|$$

Hence, since $X^{**} \to X$ and $X^* \to X$ as $\Delta u \to 0$, the upper right-derivative of $f(u)$ is at most $Mf(u)$—whence by real function theory[28] $f(0) = 0$ implies $f(u) \equiv 0$.

Theorem 10: If $t(u)$ has a continuous positive derivative, then

$$(24) \qquad \int_a^b X(t)\,dt = \int_{f(a)=a}^{f(u)=b} [X(t(u)) \cdot t'(u)]\,du$$

Proof: This is a corollary of Theorems 8–9; it is easy to show that the integral on the left has the same derivative with respect to u as the integral on the right.

16. **Invariance under group-isomorphisms.** The theory of product integration is invariant, not only under all topological *ring*-isomorphisms, but also under differentiable *group*-isomorphisms. This basic fact has been ignored in the literature; it will be established in this section.

Our ideas will be clarified if we remember

Theorem 11: Product integrals and derivatives do not depend on $|X|$ or $\|X\|$, except through the topology defined by the metric $|X - Y|$.

This is obvious. Now consider the definition of product-derivatives: $Y'(t) = \operatorname{Lim}_{\Delta t \to 0} Y^{-1}(t)Y(t + \Delta t)$. Clearly $Y'(t)$ is invariant under any one-one correspondence which preserves (a) products (and hence[29] group-inverses), and (b) $\operatorname{Lim}_{\Delta t \to 0} \dfrac{1}{\Delta t}[U(\Delta t)]$ whenever it exists. A fortiori

[28] It is easy to show for each $\lambda > 0$, that there is no first point at which $f(u) \geqq \lambda \exp(u + Mu)$. This is the only part of Riemann's theory of integration which is not a corollary of the present paper.

[29] Since 0 is the unique solution of the identity $OX = X$, and $Y^{-1}(t)$ is the unique solution Z of the equation $Y(t)Z = 0$.

Theorem 12: Product-derivatives are preserved under all group isomorphisms differentiable[30] at 0.

It follows by Theorems 9–10 that the product *integrals* of functions continuous in the metric $|X - Y|$ having bounded L-modulus, are preserved under all differentiable group-isomorphisms which carry them into similar functions. We can also show by appealing to the definition, that $\operatorname{Lim}_{\tau \to 0} J_\tau(X)$ is preserved under many group-isomorphisms.

17. **Product integration in analytical groups.** It is very easy to show that the assumptions of §§9–10 are valid in analytical groups.[31] For completeness is assumed: moreover $|AXB - AYB| \leq K \cdot |X - Y|$ in any bounded region. Finally, $|XY - (X + Y)| \leq o(|X| + |Y|)$. We conclude (since any continuous function is bounded, letting X play the rôle of L-modulus as well as D-modulus).

Theorem 13: Continuous functions in analytical groups are product integrable.

Again, the properties of product integrals proved in §12 are deduced from the assumptions of §§9–10 alone; therefore they hold also in analytical groups. Finally, since any topological isomorphism between analytical groups is (op. cit., Cor. 6.3) differentiable at 0, we have

Theorem 14: Product derivatives in analytical groups are preserved under all topological (group-) isomorphisms.

18. **Topologico-algebraic interpretation.** A goal of modern mathematical research (whose attainment may involve a deepened understanding of algebra) is the topological interpretation of metric considerations.

This raises the question: how much algebra is needed to define product derivatives and integrals topologico-algebraically? The definitions used above require merely that group products XY, a topology,

[30] In the usual sense. With Lie groups, this means the functions expressing the new coordinates in terms of the old have first partial derivatives and a non-vanishing Jacobian. In general (the parameter-space a Banach space), Fréchet's definition ("La notion de differentielles dans l'analyse générale", Ann. de l'Ec. Norm. Sup. (3) 42 (1925), 293–323) has the needed property (b).

[31] In the sense of the author's article "Analytical groups", to appear in the Trans. Am. Math. Soc. It is obvious that any Lie group has the properties assumed.

and scalar products X be defined. Moreover, in the case of analytical groups one can define scalar products in terms of group products and the topology: X^n is $XX \cdots X$ (n times), $\frac{1}{2}X$ is (near 0) the unique solution of $Y^2 = X$, and so λX can be defined in general by dyadic approximation to λ.

Another problem is to find topologico-algebraic conditions which make all (say) continuous functions integrable. We shall not solve this problem, but shall suggest some helpful methods.

The proper topologico-algebraic interpretation of completeness in the case of metrizable groups[32] was given by van Dantzig:[33] a topological group is complete if and only if every "fundamental" sequence $\{x_n\}$ such that $x_m^{-1}x_n \to 0$ as $m, n \to \infty$, approaches an actual limit x.

We can also interpret locally uniform continuity in topological groups. The condition

$$(25) \qquad |AXB - AYB| \leqq (1 + M)\cdot|X - Y| \quad [A, B \in \Omega_R^M]$$

is equivalent to: to every neighborhood $\mathfrak{V}$ of 0 there corresponds a neighborhood $\mathfrak{U} = 1/n\ \mathfrak{V}$ of 0 so small that $X^{-1}Y \in \mathfrak{U}$ implies $(AXB)^{-1}(AYB) \in \mathfrak{V}$. But this means that the set-union of the $B^{-1}\mathfrak{U}B (B \in \Omega_R^M)$ is in $\mathfrak{V}$. And this means that every $B\mathfrak{V}B^{-1}$ must contain $\mathfrak{U}$. Therefore

Theorem 15: Multiplication in a topological group is locally uniformly continuous if and only if some neighborhood $\mathfrak{V}$ of 0 is so small that the set-product of the conjugates $B^{-1}\mathfrak{U}B$ of *any* neighborhood of 0, relative to the B in $\mathfrak{V}$, contains 0 in its interior.

Remark 1: Locally uniform continuity implies "completability" in the sense of van Dantzig. Moreover, multiplication in any *locally compact* group is locally uniformly continuous. These assertions will not be proved; incidentally van Dantzig has already shown that any locally compact group is complete.

Remark 2: If multiplication is uniformly continuous in $\mathfrak{V}$, it is uniformly continuous in *every* bounded region (bounded in the sense that it is contained in some power $\mathfrak{V}\mathfrak{V} \cdots \mathfrak{V}$ of $\mathfrak{V}$).

This leaves open the problem of finding a satisfactory topologico-

[32] I.e., of groups satisfying Hausdorff's first countability axiom—as has been shown by Kakutani and the author.

[33] "Zur topologische Algebra. I. Komplettierungstheorie", Math. Annalen 107 (1933), 587–626. The criterion can be extended to arbitrary topological groups; cf. the author's "Moore-Smith convergence in general topology", Annals of Math. 38 (1937), esp. p. 48.

algebraic interpretation to the metric condition $|(XY) - (X + Y)| = o(|X| + |Y|)$, involving nothing but group properties. The reader should be cautioned that in order for a topologico-algebraic interpretation of product integration to be satisfactory, it must explain for which topological groups, and why, product integrals are preserved under topological group-isomorphisms.

19. **Application to differential equations.** Theorems 4 and 8 yield as corollaries, generalizations[34] of classical theorems on the existence of solutions of systems of ordinary differential equations.

To see this, observe first that if $dY/dt = X(t)$, and if $Y(x; t)$ denotes the transform of x under $Y(t)$, then

$$\partial/\partial t[Y(x; t)] = X(Y(x; t); t) \text{ for all } x, t. \tag{26}$$

Proof: $Y(x; t + \Delta t) - Y(x; t)$ is the displacement of $Y(x; t)$ under $Y^{-1}(t)Y(t + \Delta t)$. And this vector, divided by Δt, approaches $X(t)$ as $\Delta t \to 0$ by definition of derivative.]

Again, the hypothesis that $\|X(t)\|$ is bounded is simply the hypothesis that $X(t)$ satisfies a uniform Lipschitz condition in the usual[35] sense. While the continuity of $X(t)$ in the metric $|X - Y|$ is the assumption that $X(x; t + \Delta t)$ converges *uniformly* to $X(x; t)$ as $\Delta t \to 0$. We conclude from Theorems 4 and 8,

Theorem 16: If $X(x; t)$ is *uniformly* continuous in t and satisfies a Lipschitz condition, then the equation $\partial/\partial t[Y(x; t)] = X(Y(x; t); t)$ has a solution.

Corollary: For each $x_0 = Y(x; 0)$, we have a solution of $dx/dt = X(x; t)$ which also satisfies $x(0) = x_0$.

Again, Theorem 9 yields a uniqueness theorem on *complete systems* of solutions to $dx_i/dt = X_i(x; t)$, but it does not show that there may not be several individual solutions all having the same $x(0)$—although this is well-known to be impossible if a Lipschitz condition holds.

The application of the Lebesgue theory of Chapter III to systems of ordinary differential equations yields by similar arguments *improved* existence and uniqueness theorems which we shall not however state.

[34] Generalizations from Euclidean space to Banach space. These results are not related to papers of L. M. Graves (Trans. Am. Math. Soc. 29 (1927), 514–42) and A. D. Michal and V. Elconin (Proc. Nat. Acad. Sci. 21 (1935), 534–6), although the latter also treat differential equations involving Banach spaces.

[35] Cf. G. D. Birkhoff, "Dynamical systems", p. 5.

Digression: local hypotheses. One can easily extend the above results to the case where $X(t)$ is defined only *locally*. To see this, suppose $X(t)$ is defined only in the sphere $|p| \leqq 1$. Extend $X = X(t)$ to the shell $|q| > 1$ through the equation

$$qX - q = (q/|q|)X - (q/|q|) \tag{27}$$

i.e., by *similarity* from the surface of the sphere. Then the algebra is preserved, $|X(t)|$ is unchanged, and $||X||$ is increased in a ratio at most two.[36] Hence we can apply Theorems 4–10 to the extended $X(t)$.

To apply this method[37] of extension, note that $X(t)$ is defined throughout *some* closed sphere about every point interior to its domain of definition.

20. **Application to dependent probabilities.** The infinitesimal calculus of (linear) transformations is also fundamental in the theory of dependent probabilities.

To see this, suppose γ is any system whose transitions occur in occasional jumps. Such a system might be an atom, bombarded by occasional cosmic rays.

If γ can occupy only a finite number of states, then the transition probability can be represented by a *matrix* $P(t, u) = ||p_{ij}(t, u)||$, whose coefficients express the numerical probability that γ when in state i at the instant t, will be found in state j at the instant u. Moreover, provided successive transition probabilities are *independent*,[38] $t < u < v$ implies $p_{ij}(t, v) = \sum_k p_{ik}(t, u)p_{kj}(u, v)$. In the notation of matrix algebra, $P(t, v) = P(t, u)P(u, v)$.

Consequently if $P(O, t) = P(t)$ has a group-inverse (and this is the case for systems whose transitions occur by occasional jumps), then $P(t, u) = P^{-1}(t)P(u)$. And so[39] the *rate of transition* $\mathrm{Lim}_{\Delta t \to 0} \frac{1}{\Delta t}[P(t, t + \Delta t) - 1]$ of γ, as usually understood, is the *product derivative* of the matrix-function $p(t)$.

[36] This depends on the lemma: if $|p| = |q| = 1$, and $0 \leqq \lambda \leqq 1$, then $|p - \lambda q| \geqq \frac{1}{2}|p - q|$. We shall omit details.

[37] There are undoubtedly other methods—potential theory suggests defining $X_1(x; t)$ as the potential at x caused by the potential $X_1(x; t)$ on the surface of the domain of definition.

[38] This is the definition of *independent* probabilities. For the theory of dependent probabilities, the reader is referred to Kolmogoroff, Math. Annalen, vol. 104 (1931), p. 415.

[39] Caution: $P^{-1}(t)$ need *not* be the matrix of probabilities from effect to cause.

These considerations extend to systems having *any* Boolean algebra of qualities—except that $P(t)$ is a bounded linear operator on a Banach space, instead of being a finite matrix.[40] But in the case of systems whose state is changing *continuously*, $P(t)$ becomes unbounded—and so the theory breaks down.

From Theorems 4 and 8, we conclude as a corollary (since the bounded linear operators on any Banach space are an analytical group),

Theorem 17: Any continuously varying rate of transition is associated with a (unique) one-parameter family of transition probabilities.

Further, since every linear operator near I has exactly one square root, and since dyadic approximation yields in the limit a unique one-parameter subgroup generated by a bounded infinitesimal linear operator Q, we obtain

Theorem 18: If $P(t, u)$ depends only on $u - t$, and if $P(t, u)$ is continuous, then $P(t, u) = \exp[(u - t)Q]$ has a constant derivative Q (temporally homogeneous case).

III. Lebesgue Integration

21. **Relation to Riemann integral.** A Lebesgue product integral can be obtained from the Riemann integral defined in §6 by making four changes. To see this, recall the conclusions of Chapter II concerning continuous functions with bounded L-modulus.

It was shown (cf. §§9–10) that as $|\pi| \to 0$, the products

$$\prod_{k=1}^{n}\left[\sum_{\alpha} X(t_k^{\alpha})\cdot|\Delta_k^{\alpha}|\right] \quad [t_k^{\alpha} \epsilon \Delta_k]$$

converged to $J(X)$. (Recall that the Δ_k^{α} are subsegments, finite in number, of segments Δ_k of the domain $[a, b]$ of integration.)

Suppose we consider instead—in the spirit of Lebesgue—the wider class of products

$$\sum_{k=1}^{n}\left[\sum_{\alpha} X(t_k^{\alpha})\cdot|\sigma_k^{\alpha}|\right] \quad [t_k^{\alpha} \epsilon \sigma_k^{\alpha}] \tag{28}$$

where (a) the subsets σ_k^{α} are any *measurable*[41] sets, and (b) they may be *countable* in number.

And suppose—to make convergence as likely as possible—we (c)

[40] For the details, cf. the author's article "On dependent probabilities", to appear in the Annals of Mathematics.

[41] In the sense of Borel-Lebesgue; we denote by $|\sigma|$ the *measure* of σ.

require $t_k^\alpha \epsilon \sigma_k^\alpha$, and (d) consider the σ_k^α as *intrinsic* to a partition—i.e., admit as *subpartitions* of a partition π into Δ_k and then (within each Δ_k) σ_k^α, only partitions subsegments Δ_i' of the Δ_k and then (within each Δ_k') subsets $\sigma_j'^\beta$ of the σ_k^α.

That is, suppose we deal with "double partitions"—first into segments $\Delta_1, \cdots, \Delta_n$, and then of each Δ_k into finite or countable measurable subsets σ_k^α. And suppose we introduce as above, the notion of a *subpartition* of a double partition. Then since any two double partitions π: $[a, b] \to \Delta \to \sigma_k^\alpha$ and π^*: $[a, b] \to \Delta_k^* \to \sigma_k^{*\alpha}$ have a "product" $\pi\pi^*$: $[a, b] \to \Delta_i\Delta_j^* \to \sigma_i^\alpha\, \sigma_j^{*\beta}$ which is a[42] subpartition of both, the key consideration of §9 applies.

22. **Trivial properties.** These properties of double partitions indicate that at least certain crude properties hold under the following

Definition: $X(t)$ is L-integrable on $[a, b]$ if and only if to each neighborhood $\mathfrak{U}$ of some $J(X)$ (called the "L-integral" of $X(t)$ on $[a, b]$) corresponds a double partition $\pi(\mathfrak{U})$ of $[a, b]$ such that $\pi \leqq \pi(\mathfrak{U})$ implies

$$\prod_{k=1}^{n}\left[\sum_\alpha X(t_k^\alpha)\cdot|\sigma_k^\alpha|\right] \epsilon\, \mathfrak{U}[t_k^\alpha \epsilon \sigma_k^\alpha] \tag{29}$$

For example, it shows that.

Theorem 19: No $X(t)$ has more than one integral.

Proof: Suppose J and J' were distinct integrals of $X(t)$ over $[a, b]$. They would have disjoint neighborhoods $\mathfrak{U}$ and $\mathfrak{U}'$—and the product partition $\pi(\mathfrak{U})\pi(\mathfrak{U}')$ would yield through (29) elements in $\mathfrak{U}$ and $\mathfrak{U}'$ simultaneously—which is impossible since $\mathfrak{U} \cap \mathfrak{U}' = 0$.

Theorem 20: If $a \leqq b \leqq c$, and if $X(t)$ is L-integrable on $[a, b]$ to J and on $[b, c]$ to J', then it is L-integrable on $[a, c]$ to JJ'.

Proof: Identical with that of Theorem 6.

Finally, it is obvious that

Theorem 21: The function $X(t) = 0$ is L-integrable to 0 on any $[a, b]$. Also, Lebesgue integrals are invariant under rigid translations $t \to a + t$ of the domain of integration.

The above results are extremely general:[43] they hold provided the

[42] By $\Delta_i\Delta_j^*$ we mean the non-void set-products of the Δ_i and Δ_j^*, and similarly with the $\sigma_i^\alpha\sigma_j^\beta$. The $\Delta_i\Delta_j^*$ are disjoint segments of $[a, b]$, and the $\sigma_i^\alpha\sigma_j^\beta$ are disjoint measurable subsets of the $\Delta_i\Delta_j^*$.

[43] So is the definition of integrability: cf. §28.

values of $X(t)$ lie in a linear space in which a topology and a multiplication continuous in that topology, are defined.

23. **Further desiderata.** But there are other properties which one expects from a product integral, which cannot be established so easily. What can we say about

(30a) The L-integrability of even constant functions: (This depends on the properties listed in §§9–10: on *completeness*, on $x \circ x$ being nearly $x + x$ near 0, and on multiplication being *uniformly* continuous.)

(30b) Reciprocity of L-integrals with derivatives (i.e., relation between definite and indefinite integrals?) It is conceivable that there exist two infinite sequences of functions $X_k(t)$ and $Y_k(t)$, such that $Y_k(u) = \int_0^u X_k(t)dt$ and $X_{k+1}(t) = d/dt[Y_k(t)]$ identically, and yet such that the successive $Y_k(t)$ are unrelated.

(30c) Consistency with Riemann integration? How do we know that R-integrable functions are L-integrable? Or that when they are L-integrable, their L-integral coincides with their R-integral?

(30d) Invariance of L-integrals under analytical group-isomorphisms —i.e., is L-integration independent of the choice of an (extrinsic[44]) coordinate system?

24. **Functions of class** (AL). We shall establish properties (30a)-(30d), but for a restricted class[45] of functions, functions "of class (AL)"—whose definition is suggested by the ideas of §§9–10.

Indeed, a function $X(t)$, the space of whose values is complete, is L-integrable provided the differences

$$\prod_{k=1}^{N}\left[\sum_{\alpha} X(t_k^{\alpha})\cdot|\sigma_k^{\alpha}|\right] - \prod_{j=1}^{n}\left[\sum_{\beta} X(t_j'^{\beta})\cdot|\sigma_j'^{\beta}|\right] \tag{31}$$

are small for all subpartitions π' of suitable double partitions π. And if multiplication is uniformly continuous, this difference is small provided the sum of the differences[46]

[44] If Ω is an analytical group, then canonical parameters give one an *intrinsic* coordinate system (linearization), and so this difficulty can be disposed of. Also, with linear spaces, we have canonical parameters to start with, and again it is absent.

[45] Which, however, includes all Lebesgue integrable real functions!

[46] Explanation of notation: $\Delta'_{\alpha(k)}, \cdots, \Delta'_{\omega^*(k)}$ are the subsegments into which π' divides Δ_k.

$$(32) \qquad \sum_{\alpha} X(t_k^{\alpha}) \cdot |\sigma_k^{\alpha}| - \prod_{j=h(k)}^{h^*(k)} \left[\sum_{\beta} X(t_j'^{\beta}) \cdot |\sigma_j'^{\beta}| \right]$$

is small. And the differences (31) can be divided into two parts,

$$(32') \qquad \sum_{\substack{\alpha \\ h+(k)}} X(t_k^{\alpha}) \cdot |\sigma_k^{\alpha}| - \sum_{j} \left[\sum_{\beta} X(t_j'^{\beta}) \cdot |\sigma_j'^{\beta}| \right]$$

$$(32'') \qquad \sum_{j=h(k)}^{h^*(k)} Z_j - \prod_{j=h(k)}^{h^*(k)} Z_j \qquad \left[Z_j = \sum_{\beta} X(t_j'^{\beta}) \cdot |\sigma_j'^{\beta}| \right]$$

The sum of the (32′) is small (when the topology is convex) provided the oscillation of $X(t_k^{\alpha})$ on each σ_k^{α} is small. And the sum of the (32″) is small if sums and products differ by comparatively little on each Δ_k.

Now suppose the $X(t)$ are in some Ω_B'. To insure uniform continuity, we only need

$$(33a) \qquad \sup \sum_{\alpha,k} \| X(t_k^{\alpha}) \| \cdot |\sigma_k^{\alpha}| = K < +\infty$$

for some double partition π_a. For if (33a) holds, then by the convexity of $\| X \|$ it holds for all subpartitions of π_a, while by (16′) and the lower semi-continuity of $\| X \|$, $| AXB - AYB | \leq e^K \cdot | X - Y |$ for all products (29).

Again, to insure the absolute convergence of all sums, we need

$$(33b) \qquad \sup \sum_{\alpha,k} | X(t_k^{\alpha}) | \cdot |\sigma_k^{\alpha}| = K^* < +\infty$$

for some double partition π_b. If (33b) holds, then by the convexity of $| X |$, it holds for all subpartitions of π_b.

Finally, to insure that (32′) is small, we need, for some double partition $\pi(\epsilon)$ and any $\epsilon > 0$,

$$(33c) \qquad \sum_{\alpha,k} | \operatorname{osc} X(t_k^{\alpha}) | \cdot |\sigma_k^{\alpha}| < \epsilon$$

Definition: By a function "of class (AL)", will be meant one satisfying (33a)–(33c).

Theorem 22: Any function of class (AL) is Lebesgue product integrable.

Proof: Denote by π^* the product $\pi_a \pi_b \pi_c$; (33a)–(33c) will hold on π^* and on all its subpartitions π. For *some* such π and all *its* subpartitions π',

$$(34) \qquad \sum_{\alpha} \| X(t_k^{\alpha}) \| \cdot |\sigma_k^{\alpha}| < \operatorname{Log}(1 + \epsilon) \text{ on each } \Delta_k$$

Now consider the crucial differences (32′)–(32″); since $|AXB - AYB| \leqq e^K \cdot |X - Y|$ (uniform continuity) we need only show that the sum of their absolute values is small. By (33c), the sum of the (32′) is at most ϵ. And by (18*), (33b), and (34), the sum of the (32″) is at most ϵK^*. This completes the proof.

25. **Properties of the integral.** So much for (30a); we can also show that functions of class (AL) satisfy (30b)–(30c). For example

Theorem 23: If a function of class (AL) is R-integrable, its R-integral $J_R(X)$ equals its L-integral $J_L(X)$.

Proof: Every double partition is in the first instance a "Riemann" partition into segments (this is the key); hence by taking a product of suitable partitions we get a double partition π such that for all $\pi' \leqq \pi$, any $\prod_{k=1}^{n} X(t_k) \cdot |\Delta_k|$ is within ϵ of $J_R(X)$, and any $\prod_{k=1}^{n} \left[\sum_{\alpha} X(t_k^\alpha) \cdot |\sigma_k^\alpha| \right] = \prod_{k=1}^{n} Z_k$ is within ϵ of $J_L(X)$.

Moreover, we can approximate arbitrarily closely to the Z_k by *finite* sums, to these, using (18*) and a suitable π', by the corresponding finite *products*. And these[47] are $\prod_{k=1}^{r} X(t_k) \cdot |\Delta_k|$ for some $\pi'' \leqq \pi' \leqq \pi$. In summary, we can find ϵ-approximations to $J_R(X)$ and $J_L(X)$ which are within ϵ of each other, from which the conclusion follows immediately.

Theorem 24: If $X(t)$ is of class (AL) on $[a, b]$, then $W(t) = -X(a + b - t)$ is of class (AL) on $[a, b]$, while $\int_a^b X(t)\,dt$ and $\int_a^b W(t)\,dt$ are inverse to each other.

The proof is identical with that of Theorem 7; the only comment which is needed being the comment that

$$\sum_{\alpha} X(t_k^\alpha) \cdot |\sigma_k^\alpha| = -\sum_{\alpha} W(a + b - t_k^\alpha) \cdot |\sigma_k^\alpha| \quad \text{for all } k$$

Theorem 25: If $X(t)$ is of class (AL), then its L-integral is differ-

[47] To get π'' explicitly, chose π' so that the $\sum_{\alpha} |X(t_k^\alpha)| \cdot |\Delta_k|$ are small for all k. Then use the purely geometrical lemma: each partition of an interval into a finite number of measurable subsets has a subpartition π'' whose subsets are in one-one correspondence with disjoint subsets, where corresponding sets (1) are of the same measure, and (2) overlap.

entiable almost everywhere to $X(t)$, is "absolutely continuous", and of bounded Lipschitz modulus.[48]

Proof of first assertion: In virtue of Theorems 24 and 20, we need (cf. the proof of Theorem 8) only show that $\frac{1}{\delta}\int_c^{c+\delta} X(t)\,dt \to X(c)$ for almost all c on its domain. The key to the proof of this is Lebesgue's density theorem (S. Saks, L'Integrale p. 55): each σ_k^n in (33a)–(33c) has density one at almost all its points. But now by (33a), given $\epsilon^* > 0$, we can ignore all but a finite number of σ_k^n, and the effect on the difference quotients $\frac{1}{\delta}\int_c^{c+\delta} X(t)\,dt$ will, except in a set of measure $\leqq \sqrt{\epsilon^*}$, be at most $2\sqrt{\epsilon^*}$. As for the remaining σ_k^n, by the density theorem and the fact that by (33a) $|\,X(t)\,|$ is bounded on each σ_k^n, we see that whenever c is a density point, we need consider only contributions from $X(t_k^n)$. And by (33c) and (18*), except on σ_k^n of measure $\leqq [\epsilon \cdot e^{K*}/(b-a)]^{\frac{1}{2}}$, these make the difference quotient vary from $X(c)$ by $\leqq 2[\epsilon \cdot e^{K*}/(b-a)]^{\frac{1}{2}}$.

Proof of second assertion: By "absolute continuity", we mean that if $Y(u) = X(t)\,dt$ is regarded as a path in Ω_B', then the sum of the lengths of the images in Ω_B' of disjoint intervals of $[a, b]$ having a total length η, tends to zero uniformly with η. By Tonelli's classic lemma on the lower semi-continuity of length, to prove this one need only prove it for all $\prod_{k=1}^{n}\left[\sum_a X(t_k^a)\cdot|\,\sigma_k^a\,|\right]$. For these it follows from the negligibility of all but a finite number of σ_k^a in (33a)–(33c), the boundedness of $X(t)$ on the others, and (17*).

Proof of third assertion: By (33a) and (16′), $\|\,J_\pi(x)\,\| < e^K$ for all subpartitions of π_a. By the lower semi-continuity of $\|\,X\,\|$, the same is true of $\|\,J(X)\,\|$.

Theorem 26: If $Y'(t) = X(t)$ almost everywhere, if $Y(t)$ is absolutely continuous and $Y(t)$ uniformly bounded, and if $X(t)$ is of class (AL), then $Y(u) = Y(a)\circ\int_a^u X(t)\,dt$.

Proof: Set $Z(u) = Y(a)\circ\int_a^t X(t)\,dt$, and consider the real function

[48] Theorems 25–6 generalize classic theorems of Lebesgue. Cf. de la Vallée-Poussin, "Cours d'Analyse Infinitésimale", Paris, 1914, 260, 265. The generalization to functions with values in a Banach space has been effected by Bochner; cf. §28.

$F(t) = |Z(t) - Y(t)|$. Clearly $F(t)$ is absolutely continuous. And at those points t where $Y'(t) = X(t) = Z'(t)$—which by Theorem 25 and assumption means almost everywhere, we can prove as in Theorem 9 that the upper right-derivative of $F(t)$ is bounded by $MF(t)$. Hence by real function theory, since $F(a) = 0$, $F(t)$ is identically zero, proving the theorem.

26. **Invariance of theory under group-isomorphisms.** We have seen (Theorem 12) that derivatives are preserved under group-isomorphisms differentiable at 0. It follows by Theorems 25–6 that the *integrals* of functions of class (A1) are preserved under group-isomorphisms differentiable at 0.

These remarks cover (30d); as in Theorem 13, we also know that integrals of functions of class (AL) with values in an analytical group, are preserved under all topological group isomorphisms.

27. **The Denjoy integral.** One natrally asks, can the Denjoy integral as well as the Lebesgue integral be extended to functions with other than real values?

If (cf. Saks, "Theorie de l'intégrale", pp. 205, 209; §27 has meaning only when read relative to that context) one uses the definition of the Denjoy integral as the Lebesgue integral "completed" by two operations (C) and (H), one sees that (1) condition (C) can be extended to the most general product integrals, (2) condition (H) can[49] be extended without modification to functions with values in Banach spaces, but (3) in the case of non-commutative multiplication, there is no reason for supposing that the hypotheses of (H) imply the convergence of the finite products corresponding to the interpolation of finite sets of intervals I_k in their proper places, and so one has to modify (H) by saying "and *if* the finite products $\cdots$ converge, then $\cdots$".

By this means, one obtains a product integral which specializes to the Denjoy integral in the case of real functions. Only, in the commutative case, part of the definition becomes redundant.

28. **The commutative case.** A large part of the theory of integration of functions with values in a Banach space is contained in the above as the special case $\|X\| = 0$, $XY = X + Y$.

[49] Because the sums of the integral over E with those over finite sets σ of I_k converge as σ "swells" (in the terminology of Moore) provided they do for the I_k separately; this is not clear in the non-commutative case.

Relative to these definitions, any Banach space satisfies the hypotheses used above on the *range* of $X(t)$. Moreover, the *domain* of $X(t)$—which has to be a *line* except when multiplication is commutative[50]—can now be any abstract "space" with a σ-ring of measurable sets whose measures are completely additive.

Indeed, without generalizing the domain, we obtain Graves' theory of Riemann integration as a Corollary of Chapter II; generalizing the domain, we obtain most of Bochner's theory of Lebesgue integration.[51]

Finally, with the generalized domain, the definition of L-integration specializes to the author's integral[52] for functions with values in a Banach space—provided summability means *unconditional* summability (which is somewhat more general than absolute summability). It follows from known counter-examples (op. cit., p. 375) that Theorems 25–6 are not true for *all* integrable resp. differentiable functions.

Some proofs become much simpler in the case of Banach spaces; thus that of Theorem 23 does: if $X(t)$ is R-integrable, then it is bounded and the double partitions based on any Riemann partition π give one series converging absolutely to elements in the closure of the convex hull of $J_\pi(X)$.

Also, when the values of the integrand lie in a Banach space, integrals have further formal properties, which can be found in any text on integration.

For a bibliography on product integration, the reader is referred to G. Rasch, "Zur Theorie und Anwendung des Produktintegrals", Crelle's Jour. 171 (1934), 65–119.

29. **Analytical displacements of one complex variable.** Let K be the complex plane. Consider the subset Ω_4' consisting of analytic functions $zX = w(z)$ defined on $|z| < 1$.

[50] Otherwise there is no natural way to order the Δ_k, and the $\prod_{k=1}^{n} X(t_k)\cdot|\Delta_k|$ would not converge to a limit.

[51] L. M. Graves, "Riemann integration and Taylor's Theorem in general analysis", these Transactions 29 (1927), 163–77. S. Bochner, "Integration von Funktionen, deren Werte die Elemente eines Vektorraumes sind", Fund. Math. 20 (1933), 262–76. Bochner's "messbare Funktionen" are of class (AL) in our sense—and so his theorems on the reciprocity of integration and differentiation are corollaries of our Theorems 25–6. Historical note: An infinitesimal calculus of vector-valued functions was already used by Grassmann (cf. 1844 "Ausdehnungslehre", 105, and 1862 "Ausdehnungslehre", Part II, Chaps. 2–4).

[52] Cf. "On the integration of functions with values in a Banach space", Trans. Am. Math. Soc. (1935), 357–78.

Then $\| X \| = \sup | d/dz(w - z) |$. Moreover, within the subset $\| X \| = M$, the metric topology $| X - Y |$ coincides with the topology of the coefficients a_k of $w(z) = \sum_{-\infty}^{\infty} a_k z^k$. For if $| X_n - X | \to 0$, then by Cauchy's integral formulas for the derivatives of $w(z)$, the coefficients of $w_n(z)$ tend to those of $w(z)$. While the converse holds by uniform convergence and the boundedness of the a_k in every circle interior to $| z | < 1$.

It is a corollary (provable by the diagonal process) that the subsets $\| X \| \leq M$ are *compact*. Hence our theory of integration applies to these subsets—and we make an interesting contact with a type of infinite continuous group.[53]

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[53] H. Cartan, "Sur les groupes de transt. analytique", Actualité 198, Paris (1935) gives an interesting treatment which seems to point in the same direction as ours. We are using the Doppelreihensatz of Weierstrass in paragraph two of §29.

A TABLE OF PLANCK'S FUNCTION FROM 3500 TO 8000°K

By Parry Moon

The growing use, in science and engineering, of Planck's equation for the radiation from a blackbody emphasizes the need for more satisfactory tabulated values of this function. In photometry and radiometry, stress is being laid to an increasing extent upon calculations based on blackbody radiation. In colorimetry a similar movement is in progress—a movement that has been particularly marked since 1931, when the C.I.E. standard distribution coefficients were introduced. A number of tables of Planck's function are available, but all have disadvantages when values are to be obtained with the minimum of calculation.

Planck's equation is

$$J_\lambda = \frac{C_1}{\lambda^5} \frac{1}{e^{C_2/\lambda T} - 1} \tag{1}$$

where J_λ = spectral radiosity[1] of a blackbody (watts cm^{-2} $micron^{-1}$) at wavelength λ.

λ = wavelength (microns)

T = absolute temperature ($°K$)

$C_1 = 36970$ *

$C_2 = 14320$ *

Upon looking at the equation, one is struck by the possibility of making a table of single entry by introducing new variables such as

$$x = \frac{\lambda T}{C_2}, \quad y = \frac{J_\lambda \lambda^5}{C_1}$$

* According to R. T. Birge, Phys. Rev. Suppl., 1, 1929, p. 1.

[1] Parry Moon. Scientific Basis of Illuminating Engineering. McGraw-Hill, 1936, p. 546.

Tables of this nature have been given by Jahnke and Emde,[2] Frehafer and Snow,[3] Fabry,[4] and Yamauti and Okamatu.[5,6] Since there is only one independent variable, the table can be condensed into comparatively small space while still covering the entire range of wavelength and temperature. These tables, however, are anything but satisfactory. Interpolations are nearly always necessary; and the labor entailed in the interpolations, as well as in the computation of x from the known value of λ and T and the computation of J_λ from the interpolated value of y, is considerable. In fact, comparative tests showed that J_λ could be computed directly from Eq. (1) more easily than it could be obtained from the Yamauti and Okamatu[6] tables, a logarithm table and a modern calculating machine being used in both cases.

Evidently present needs require a table of double entry, with the two independent variables λ and T. Tables of this kind have been computed by Forsythe,[7] Fowle,[8] Frehafer and Snow,[9] and Skogland.[10] These tables are based on values of either 14330 or 14350 for C_2, instead of the international standard value[11] of 14320. The change to 14320 is easily made with existing tables, but requires additional computation which tends to reduce the serviceability of the tables. The Skogland results are being used extensively, both in this country and abroad, but they cover only a limited range of temperature and are confined to the visible spectrum. The Fowle table and the Frehafer and Snow table are not subject to these restrictions in temperature and wavelength but the values are in general calculated to only 2 or 3 significant figures.

A new table of Planck's function is being calculated at Massachusetts

[2] Jahnke and Emde, Funktionentafeln. Teubner, Leipzig, 1933. Plancksche Strahlungsfunktion, p. 44.

[3] Frehafer and Snow, Tables and Graphs for Facilitating Computations of Spectral Energy Distribution. Bu. Stds. Msc. Pub. No. 56, 1925, Table I.

[4] C. Fabry, Introduction générale a la photométrie. Paris, 1927, p. 121.

[5] Yamauti and Okamatu, Tables of Planck's Formula of Radiation. Res. El. Lab., Tokyo, No. 395, 1936.

[6] Res. El. Lab., Tokyo, No. 402, 1936.

[7] W. E. Forsythe, 1919 Report of Standards Committee on Pyrometry. J. O. S. A., 4, 1920, p. 331.

[8] F. E. Fowle, Radiation from a Perfect (Blackbody) Radiator. Int. Crit. Tables, V, pp. 239–241. McGraw-Hill, 1929.

[9] Frehafer and Snow, Bu. Stds. Misc. Pub. No. 56, 1925, Table II.

[10] J. F. Skogland, Tables of Spectral Energy Distribution. Bu. Stds. Misc. Pub., No. 86, 1929.

[11] G. K. Burgess, The International Temperature Scale. Bu. Stds. J. R., 1, 1928, p. 635.

TABLE I

T	C_1/T		T	C_1/T		T	C_1/T	
2000°K	3.1095	48491	2350°K	2.6464	24247	2700°K	2.3033	69252
2010	3.0940	78100	2360	2.6352	10585	2710	2.2948	69734
2020	3.0787	60881	2370	2.6240	91553	2720	2.2864	32713
2030	3.0635	94571	2380	2.6130	65958	2730	2.2780	57502
2040	3.0485	76951	2390	2.6021	32628	2740	2.2697	43423
2050	3.0337	05844	2400	2.5912	90409	2750	2.2614	89811
2060	3.0189	79116	2410	2.5805	38166	2760	2.2532	96007
2070	3.0043	94676	2420	2.5698	74785	2770	2.2451	61364
2080	2.9899	50471	2430	2.5592	99169	2780	2.2370	85244
2090	2.9756	44488	2440	2.5488	10238	2790	2.2290	67018
2100	2.9614	74753	2450	2.5384	06931	2800	2.2211	06065
2110	2.9474	39327	2460	2.5280	88203	2810	2.2132	01772
2120	2.9335	36311	2470	2.5178	53028	2820	2.2053	53539
2130	2.9197	63840	2480	2.5077	00395	2830	2.1975	60770
2140	2.9061	20084	2490	2.4976	29309	2840	2.1898	22880
2150	2.8926	03246	2500	2.4876	38792	2850	2.1821	39291
2160	2.8792	11565	2510	2.4777	27880	2860	2.1745	09433
2170	2.8659	43309	2520	2.4678	95627	2870	2.1669	32745
2180	2.8527	96780	2530	2.4581	41099	2880	2.1594	08673
2190	2.8397	70310	2540	2.4484	63378	2890	2.1519	36671
2200	2.8268	62264	2550	2.4388	61561	2900	2.1445	16200
2210	2.8140	71032	2560	2.4293	34758	2910	2.1371	46728
2220	2.8013	95036	2570	2.4198	82093	2920	2.1298	27733
2230	2.7888	32726	2580	2.4105	02705	2930	2.1225	58696
2240	2.7763	82580	2590	2.4011	95745	2940	2.1153	39109
2250	2.7640	43102	2600	2.3919	60377	2950	2.1081	68468
2260	2.7518	12823	2610	2.3827	95778	2960	2.1010	46277
2270	2.7396	90300	2620	2.3737	01137	2970	2.0939	72047
2280	2.7276	74114	2630	2.3646	75658	2980	2.0869	45295
2290	2.7157	62873	2640	2.3557	18553	2990	2.0799	65545
2300	2.7039	55209	2650	2.3468	29049	3000	2.0730	32327
2310	2.6922	49775	2660	2.3380	06383	3010	2.0661	45176
2320	2.6806	45250	2670	2.3292	49805	3020	2.0593	03636
2330	2.6691	40335	2680	2.3205	58575	3030	2.0525	07254
2340	2.6577	33752	2690	2.3119	31963	3040	2.0457	55585
2350	2.6464	24247	2700	2.3033	69252	3050	2.0390	48190

TABLE I—*Continued*

T	C_1/T		T	C_1/T		T	C_1/T	
3050°K	2.0390	48190	3400°K	1.8291	46171	6000°K	1.0365	16163
3060	2.0323	84634	3410	1.8237	82105	6100	1.0195	24095
3070	2.0257	64488	3420	1.8184	49409	6200	1.0030	80158
3080	2.0191	87331	3430	1.8131	47807	6300	0.9871	58251
3090	2.0126	52744	3440	1.8078	77029	6400	0.9717	33903
3100	2.0061	60316	3450	1.8026	36806	6500	0.9567	84151
3110	1.9997	09640	3460	1.7974	26873	6600	0.9422	87421
3120	1.9933	00314	3470	1.7922	46968	6700	0.9282	23430
3130	1.9869	31942	3480	1.7870	96833	6800	0.9145	73085
3140	1.9806	04134	3490	1.7819	76212	6900	0.9013	18403
3150	1.9743	16501	3500	1.7768	84852	7000	0.8884	42426
3160	1.9680	68664	3600	1.7275	26939	7100	0.8759	29152
3170	1.9618	60246	3700	1.6808	37022	7200	0.8637	63469
3180	1.9556	90874	3800	1.6366	04469	7300	0.8519	31093
3190	1.9495	60182	3900	1.5946	40252	7400	0.8404	18511
3200	1.9434	67807	4000	1.5547	74245	7500	0.8292	12931
3210	1.9374	13389	4100	1.5168	52922	7600	0.8183	02234
3220	1.9313	96577	4200	1.4807	37376	7700	0.8076	74033
3230	1.9254	17021	4300	1.4463	01623	7800	0.7973	20126
3240	1.9194	74376	4400	1.4134	31132	7900	0.7872	27466
3250	1.9135	68301	4500	1.3820	21551	8000	0.7773	87123
3260	1.9076	98460	4600	1.3519	77604	8100	0.7677	89751
3270	1.9018	64520	4700	1.3232	12123	8200	0.7584	26461
3280	1.8960	66152	4800	1.2956	45204	8300	0.7492	88793
3290	1.8903	03033	4900	1.2692	03465	8400	0.7403	68688
3300	1.8845	74843	5000	1.2438	19396	8500	0.7316	58468
3310	1.8788	81263	5100	1.2194	30780	8600	0.7231	50812
3320	1.8732	21982	5200	1.1959	80188	8700	0.7148	38733
3330	1.8675	96690	5300	1.1734	14524	8800	0.7067	15566
3340	1.8620	05084	5400	1.1516	84626	8900	0.6987	74942
3350	1.8564	46860	5500	1.1307	44905	9000	0.6910	10776
3360	1.8509	21720	5600	1.1105	53032	9100	0.6834	17251
3370	1.8454	29371	5700	1.0910	69645	9200	0.6759	88802
3380	1.8399	69521	5800	1.0722	58100	9300	0.6687	20105
3390	1.8345	41882	5900	1.0540	84234	9400	0.6616	06062
3400	1.8291	46171	6000	1.0365	16163	9500	0.6546	41787

TABLE I—*Concluded*

T	C_2'/T	T	C_2'/T	T	C_2'/T
9500°K	0.6546 41787	10000°K	0.6219 09698	15000°K	0.4146 06465
9600	0.6478 22602	11000	0.5653 72453	16000	0.3886 93561
9700	0.6411 44019	12000	0.5182 58082	17000	0.3658 29234
9800	0.6346 01733	13000	0.4783 92075	18000	0.3455 05388
9900	0.6281 91614	14000	0.4442 21213	19000	0.3273 20894
10000	0.6219 09698	15000	0.4146 06465	20000	0.3109 54849

Institute of Technology for temperatures from 2000 to 8000°K, 10-degree intervals being employed from 2000 to 3500 and 100-degree intervals from 3500 to 8000. The portion of the table from 3500 to 8000°K has been completed, and it was thought advisable to publish this section now rather than to await the completion of the project. Most of the computation was done by students paid by the National Youth Administration.

Actual computations were made by use of the equation:

$$J_\lambda = \frac{36970}{\lambda^5}\left[10^{\frac{6219.096981}{\lambda T}} - 1\right]^{-1} \text{(watts cm}^{-2}\text{ micron}^{-1}\text{)} \ldots\ldots (2)$$

The first step was to calculate values of C_2'/T for all temperatures likely to be used in practice. Here $C_2' = 0.4342944819\ C_2 = 6219.096981$. The use of these values (Table I) will expedite blackbody calculations when such must be made outside the range of the available tables. A second auxiliary table (Table II) gives values of C_1/λ^5.

In using Eq. (2), the computers first divided values from Table I by λ. The powers of 10 were then read directly from Peter's *Zehnstellige Logarithmen*,[12] and the multiplications and divisions were made on electrically driven calculating machines. Results were tabulated to seven significant figures and were checked by use of fifth differences. After check, the seven-figure results were rounded off to five figures. All the printed numbers have been compared with the original computation books. It is believed that in general the values of Table III are correct in the last figure, though because of the rounding process there may be an error in a few instances of as much as 5 in the sixth figure.

[12] J. Peters, Zehnstellige Logarithmen der Zahlen von 1 bis 100,000. Berlin, 1922, Bd. I.

TABLE II

λ (microns)	C_1/λ^5		λ (microns)	C_1/λ^5		λ (microns)	C_1/λ^5	
0.26	31115	924.62	0.60	47543	7.2427	0.95	47778.	38042
0.27	25765	028.65	0.61	43772	3.9157	0.96	45341.	22875
0.28	21481	237.37	0.62	40354	4.1009	0.97	43051.	74698
0.29	18024	343.74	0.63	37251	7.5201	0.98	40899.	60108
			0.64	34430	9.9558	0.99	38875.	27530
0.30	15213	991.76						
0.31	12913	411.24	0.65	31862	7.0682	1.00	36970.	00000
0.32	11017	918.59	0.66	29520	9.1219	1.10	22955.	46131
0.33	94466	91.897	0.67	27382	6.4759	1.20	14857.	41383
0.34	81368	22.938	0.68	25427	5.7167	1.30	9957.0	95880
			0.69	23637	6.3362	1.40	6873.9	95953
0.35	70389	71.854						
0.36	61141	62.071	0.70	21996	7.8705	1.50	4868.4	77364
0.37	53313	94.629	0.71	20490	7.4303	1.60	3525.7	33947
0.38	46658	57.465	0.72	19106	7.5647	1.70	2603.7	83338
0.39	40975	70.322	0.73	17833	4.4102	1.80	1956.5	31863
			0.74	16660	6.0821	1.90	1493.0	74388
0.40	36103	51.562						
0.41	31910	24.708	0.75	15579	1.2757	2.00	1155.3	12500
0.42	28288	04.919	0.76	14580	8.0457	2.10	905.21	75746
0.43	25148	21.546	0.77	13658	2.7412	2.20	717.35	81662
0.44	22417	44.269	0.78	12804	9.0725	2.30	574.39	44969
			0.79	12014	7.2940	2.40	464.29	41822
0.45	20034	88.628						
0.46	17949	82.803	0.80	11282	3.4863	2.50	378.57	28000
0.47	16119	81.899	0.81	10602	8.9244	2.60	311.15	92462
0.48	14509	19.320	0.82	99719.	52233	2.70	257.65	02865
0.49	13087	87.234	0.83	93855.	34030	2.80	214.81	23737
			0.84	88400.	15382	2.90	180.24	34374
0.50	11830	40.000						
0.51	10715	15.777	0.85	83321.	06686	3.00	152.13	99176
0.52	97237	2.6443	0.86	78588.	17333	4.00	36.103	51562
0.53	88403	6.3088	0.87	74174.	25406	5.00	11.830	40000
0.54	80515	7.1454	0.88	70054.	50840	6.00	4.7543	72427
			0.89	66206.	31520	7.00	2.1996	78705
0.55	73457	4.7620						
0.56	67128	8.6675	0.90	62609.	01966	8.00	1.1282	34863
0.57	61443	3.9045	0.91	59243.	74295			
0.58	56326	0.7416	0.92	56093.	21260			
0.59	51711	7.6874	0.93	53141.	61000			
			0.94	50374.	43436			

Table III (see next page) gives values of spectral radiosity—that is, the watts radiated into a hemisphere from a blackbody opening of one sq. cm., per micron band of wavelengths. The wavelength values are sufficiently close so that interpolation is generally unnecessary. For convenience in temperature interpolation (which is often required) values of first differences are given under Δ. Table III covers the temperature range above that of the tungsten lamp, this range being needed ordinarily in the specification of the various phases of daylight. The temperature intervals are 100° and the wavelength intervals are as follows:

0.26 to 0.75μ,	0.01μ intervals
0.75 to 1.00μ,	0.05μ intervals
1.00 to 3.00μ,	0.10μ intervals

The range of wavelengths appears to be sufficiently broad for most practical purposes. In the ultraviolet, the computations extend through the biologically important wavelengths near 0.29μ. In the infrared, they are carried out to 3μ, which is approximately the limit imposed by the transmission of glass. Though there are advantages to the use of relative J's, as was done by Skogland, the use of absolute values seemed on the whole preferable, and calculations were made on this basis.

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TABLE III

λ (microns)	3500°K		3600°K		3700°K		3800°K		3900°K		4000°K
	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ
0.26	4.5584	24991	7.0575	36140	10.671	5118	15.789	7107	22.896	9694	32.590
0.27	6.7605	35382	10.299	5037	15.336	7027	22.363	9623	31.986	12951	44.937
0.28	9.6837	48481	14.532	6802	21.334	9359	30.693	12650	43.343	16815	60.158
0.29	13.448	6453	19.901	8931	28.832	12131	40.963	16198	57.161	21285	78.446
0.30	18.167	8368	26.535	11436	37.971	15350	53.321	20262	73.583	26340	99.923
0.31	23.942	10602	34.544	14320	48.864	19005	67.869	24823	92.692	31944	124.64
0.32	30.856	13157	44.013	17574	61.587	23080	84.667	29846	114.51	3805	152.56
0.33	38.975	16024	54.999	21180	76.179	27545	103.72	3529	139.01	4458	183.59
0.34	48.342	19188	67.530	25115	92.645	32358	125.00	4109	166.09	5148	217.57
0.35	58.979	22627	81.606	29343	110.95	3747	148.42	4720	195.62	5866	254.28
0.36	70.884	26314	97.198	33827	131.03	4284	173.87	5353	227.40	6605	293.45
0.37	84.034	30217	114.25	3852	152.77	4842	201.19	6004	261.23	7356	334.79
0.38	98.385	34300	132.68	4339	176.07	5413	230.20	6665	296.85	8112	377.97
0.39	113.87	3853	152.40	4837	200.77	5992	260.69	7331	334.00	8864	422.64
0.40	130.42	4286	173.28	5344	226.72	6574	292.46	7993	372.39	9607	468.46
0.41	147.94	4726	195.20	5852	253.72	7156	325.28	8646	411.74	10333	515.07
0.42	166.32	5169	218.01	6360	281.61	7730	358.91	9285	451.76	11038	562.14
0.43	185.46	5611	241.57	6863	310.20	8291	393.11	9907	492.18	11716	609.34
0.44	205.24	6049	265.73	7356	339.29	8838	427.67	10506	532.73	12362	656.35
0.45	225.53	6481	290.34	7836	368.70	9367	462.37	11078	573.15	12974	702.89
0.46	246.22	6902	315.24	8302	398.26	9873	496.99	11621	613.20	13548	748.68
0.47	267.19	7311	340.30	8749	427.79	10354	531.33	12133	652.66	14083	793.49
0.48	288.33	7704	365.37	9176	457.13	10809	565.22	12611	691.33	14578	837.11
0.49	309.51	8082	390.33	9580	486.13	11236	598.49	13054	729.03	15030	879.33
0.50	330.64	8441	415.05	9960	514.65	11634	630.99	13461	765.60	15440	920.00
0.51	351.60	8781	439.41	10316	542.57	12001	662.58	13832	800.90	15808	958.98
0.52	372.32	9099	463.31	10647	569.78	12337	693.15	14166	834.81	16135	996.16
0.53	392.69	9397	486.66	10951	596.17	12642	722.59	14465	867.24	16419	1031.4
0.54	412.64	9673	509.37	11229	621.66	12916	750.82	14729	898.11	16663	1064.7
0.55	432.11	9925	531.36	11482	646.18	13159	777.77	14957	927.34	16869	1096.0
0.56	451.02	10156	552.58	11707	669.65	13373	803.38	15151	954.89	17037	1125.3
0.57	469.32	10365	572.97	11906	692.03	13558	827.61	15313	980.74	17169	1152.4
0.58	486.96	10552	592.48	12081	713.29	13713	850.42	15444	1004.9	1726	1177.5
0.59	503.90	10717	611.07	12231	733.38	13842	871.80	15544	1027.2	1734	1200.6
0.60	520.11	10861	628.72	12357	752.29	13944	891.73	15616	1047.9	1737	1221.6
0.61	535.56	10984	645.40	12460	770.00	14022	910.22	15660	1066.8	1738	1240.6
0.62	550.22	11088	661.10	12542	786.52	14074	927.26	15681	1084.1	1735	1257.6
0.63	564.09	11172	675.81	12603	801.84	14105	942.89	15675	1099.6	1731	1272.7
0.64	577.16	11238	689.54	12642	815.96	14115	957.11	15648	1113.6	1724	1286.0

TABLE III—*Continued*

λ (microns)	3500°K		3600°K		3700°K		3800°K		3900°K		4000°K
	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ
0.65	589.41	11286	702.27	12664	828.91	14104	969.95	15601	1126.0	1714	1297.4
0.66	600.85	11318	714.03	12667	840.70	14075	981.45	15533	1136.8	1704	1307.2
0.67	611.47	11334	724.81	12655	851.36	14028	991.64	15448	1146.1	1691	1315.2
0.68	621.30	11334	734.64	12626	860.90	13965	1000.6	1534	1154.0	1677	1321.7
0.69	630.33	11321	743.54	12583	869.37	13887	1008.2	1523	1160.5	1661	1326.6
0.70	638.57	11296	751.53	12525	876.78	13795	1014.7	1510	1165.7	1644	1330.1
0.71	646.06	11256	758.62	12456	883.18	13691	1020.1	1496	1169.7	1624	1332.1
0.72	652.79	11206	764.85	12375	888.60	13575	1024.3	1481	1172.4	1605	1332.9
0.73	658.79	11146	770.25	12283	893.08	13448	1027.6	1463	1173.9	1585	1332.4
0.74	664.08	11076	774.84	12181	896.65	13313	1029.8	1446	1174.4	1564	1330.8
0.75	668.69	10995	778.64	12071	899.35	13167	1031.0	1429	1173.9	1541	1328.0
0.80	682.20	10486	787.06	11411	901.17	12347	1024.6	1329	1157.5	1424	1299.9
0.85	682.09	9849	780.58	10637	886.95	11426	1001.2	1222	1123.4	1300	1253.4
0.90	671.36	9146	762.82	9812	860.94	10474	965.68	11134	1077.0	1179	1194.9
0.95	652.71	8423	736.94	8982	826.76	9537	922.13	10084	1023.0	1062	1129.2
1.00	628.47	7709	705.56	8178	787.34	8642	873.76	9095	964.71	9541	1060.1
1.10	570.42	6380	634.22	6711	701.33	7032	771.65	7346	845.11	7651	921.62
1.20	507.93	5236	560.29	5469	614.98	5694	671.92	5911	731.03	6122	792.25
1.30	447.07	4285	489.92	4450	534.42	4610	580.52	4762	628.14	4908	677.22
1.40	390.86	3510	425.96	3629	462.25	3742	499.67	3850	538.17	3953	577.70
1.50	340.54	2884	369.38	2969	399.07	3051	429.58	3129	460.87	3203	492.90
1.60	296.31	2379	320.10	2442	344.52	2502	369.54	2558	395.12	2612	421.24
1.70	257.86	1973	277.59	2020	297.79	2063	318.42	2105	339.47	2145	360.92
1.80	224.67	1644	241.11	1680	257.91	1712	275.03	1744	292.47	1773	310.20
1.90	196.10	1378	209.88	1405	223.93	1430	238.23	1454	252.77	1475	267.52
2.00	171.55	1162	183.17	1182	194.99	1201	207.00	1219	219.19	1236	231.55
2.10	150.45	985	160.30	1000	170.30	1015	180.45	1029	190.74	1042	201.16
2.20	132.30	839	140.69	852	149.21	862	157.83	873	166.56	883	175.39
2.30	116.67	718	123.85	728	131.13	737	138.50	746	145.96	753	153.49
2.40	103.17	619	109.36	625	115.61	633	121.94	640	128.34	645	134.79
2.50	91.497	5344	96.841	5405	102.25	546	107.71	551	113.22	556	118.78
2.60	81.368	4640	86.008	4690	90.698	4734	95.432	4777	100.21	482	105.03
2.70	72.557	4047	76.604	4086	80.690	4123	84.813	4158	88.971	4189	93.160
2.80	64.874	3543	68.417	3576	71.993	3605	75.598	3633	79.231	3660	82.891
2.90	58.155	3114	61.269	3141	64.410	3165	67.575	3189	70.764	3209	73.973
3.00	52.263	2747	55.010	2769	57.779	2790	60.569	2809	63.378	2826	66.204

TABLE III—*Continued*

λ (microns)	4000°K		4100°K		4200°K		4300°K		4400°K		4500°K
	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ
0.26	32.590	13007	45.597	17186	62.783	22387	85.170	28779	113.95	3654	150.49
0.27	44.937	17158	62.095	22398	84.493	28841	113.33	3667	150.00	4608	196.08
0.28	60.158	22015	82.173	28416	110.59	3620	146.79	4556	192.35	5669	249.04
0.29	78.446	27561	106.01	3520	141.21	4441	185.62	5535	240.97	6826	309.23
0.30	99.923	33758	133.68	4270	176.38	5336	229.74	6593	295.67	8061	376.28
0.31	124.64	4055	165.19	5082	216.01	6296	278.97	7715	356.12	9358	449.70
0.32	152.56	4786	200.42	5948	259.90	7308	332.98	8886	421.84	10697	528.81
0.33	183.59	5561	239.20	6856	307.76	8359	391.35	10089	492.24	12062	612.86
0.34	217.57	6371	281.28	7794	359.22	9436	453.58	11310	566.68	13433	701.01
0.35	254.28	7206	326.34	8752	413.86	10524	519.10	12533	644.43	14793	792.36
0.36	293.45	8056	374.01	9720	471.21	11611	587.32	13743	724.75	16129	886.04
0.37	334.79	8912	423.91	10684	530.75	12686	657.61	14930	806.91	17422	981.13
0.38	377.97	9765	475.62	11636	591.98	13738	729.36	16078	890.14	18665	1076.8
0.39	422.64	10607	528.71	12567	654.38	14757	801.95	17180	973.75	19846	1172.2
0.40	468.46	11430	582.76	13469	717.45	15733	874.78	18228	1057.1	2095	1266.6
0.41	515.07	12228	637.35	14335	780.70	16661	947.31	19213	1139.4	2199	1359.3
0.42	562.14	12994	692.08	15158	843.66	17536	1019.0	2013	1220.3	2295	1449.8
0.43	609.34	13723	746.57	15935	905.92	18352	1089.4	2098	1299.2	2381	1537.3
0.44	656.35	14413	800.48	16660	967.08	19106	1158.1	2176	1375.7	2459	1621.6
0.45	702.89	15058	853.47	17332	1026.8	1980	1224.8	2244	1449.2	2529	1702.1
0.46	748.68	15658	905.26	17949	1084.7	2043	1289.0	2307	1519.7	2590	1778.7
0.47	793.49	16210	955.59	18509	1140.7	2098	1350.5	2362	1586.7	2642	1850.9
0.48	837.11	16712	1004.2	1901	1194.3	2148	1409.1	2409	1650.0	2687	1918.7
0.49	879.33	17166	1051.0	1946	1245.6	2190	1464.6	2450	1709.6	2723	1981.9
0.50	920.00	17571	1095.7	1985	1294.2	2227	1516.9	2483	1765.2	2752	2040.4
0.51	958.98	17927	1138.3	2018	1340.1	2257	1565.8	2510	1816.8	2773	2094.1
0.52	996.16	18235	1178.5	2047	1383.2	2282	1611.4	2530	1864.4	2788	2143.2
0.53	1031.4	1850	1216.4	2070	1423.4	2301	1653.5	2544	1907.9	2796	2187.5
0.54	1064.7	1872	1251.9	2088	1460.7	2315	1692.2	2552	1947.4	2799	2227.3
0.55	1096.0	1889	1284.9	2102	1495.1	2324	1727.5	2555	1983.0	2796	2262.6
0.56	1125.3	1902	1315.5	2111	1526.6	2328	1759.4	2554	2014.8	2787	2293.5
0.57	1152.4	1912	1343.6	2116	1555.2	2328	1788.0	2547	2042.7	2774	2320.1
0.58	1177.5	1918	1369.3	2117	1581.0	2324	1813.4	2536	2067.0	2757	2342.7
0.59	1200.6	1920	1392.6	2114	1604.0	2316	1835.6	2522	2087.8	2735	2361.3
0.60	1221.6	1919	1413.5	2108	1624.3	2304	1854.7	2505	2105.2	2710	2376.2
0.61	1240.6	1915	1432.1	2100	1642.1	2289	1871.0	2483	2119.3	2683	2387.6
0.62	1257.6	1909	1448.5	2087	1657.2	2272	1884.4	2459	2130.3	2652	2395.5
0.63	1272.7	1900	1462.7	2073	1670.0	2251	1895.1	2433	2138.4	2618	2400.2
0.64	1286.0	1888	1474.8	2056	1680.4	2229	1903.3	2404	2143.7	2582	2401.9

TABLE III—*Continued*

λ (microns)	4000°K		4100°K		4200°K		4300°K		4400°K		4500°K
	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ
0.65	1297.4	1874	1484.8	2038	1688.6	2204	1909.0	2373	2146.3	2545	2400.8
0.66	1307.2	1858	1493.0	2016	1694.6	2178	1912.4	2340	2146.4	2506	2397.0
0.67	1315.2	1841	1499.3	1994	1698.7	2149	1913.6	2306	2144.2	2465	2390.7
0.68	1321.7	1821	1503.8	1970	1700.8	2119	1912.7	2271	2139.8	2424	2382.2
0.69	1326.6	1801	1506.7	1944	1701.1	2088	1909.9	2234	2133.3	2381	2371.4
0.70	1330.1	1779	1508.0	1917	1699.7	2056	1905.3	2196	2124.9	2338	2358.7
0.71	1332.1	1756	1507.7	1890	1696.7	2023	1899.0	2158	2114.8	2294	2344.2
0.72	1332.9	1732	1506.1	1860	1692.1	1990	1891.1	2119	2103.0	2249	2327.9
0.73	1332.4	1707	1503.1	1831	1686.2	1955	1881.7	2080	2089.7	2205	2310.2
0.74	1330.8	1681	1498.9	1801	1679.0	1920	1871.0	2040	2075.0	2160	2291.0
0.75	1328.0	1655	1493.5	1770	1670.5	1886	1859.1	2000	2059.1	2115	2270.6
0.80	1299.9	1518	1451.7	1613	1613.0	1708	1783.8	1800	1963.8	1894	2153.2
0.85	1253.4	1379	1391.3	1456	1536.9	1533	1690.2	1609	1851.1	1684	2019.5
0.90	1194.9	1243	1319.2	1306	1449.8	1369	1586.7	1431	1729.8	1491	1878.9
0.95	1129.2	1115	1240.7	1168	1357.5	1218	1479.3	1268	1606.1	1316	1737.7
1.00	1060.1	998	1159.9	1040	1263.9	1082	1372.1	1122	1484.3	1161	1600.4
1.10	921.62	7946	1001.1	823	1083.4	851	1168.5	878	1256.3	904	1346.7
1.20	792.25	6323	855.48	6520	920.68	6707	987.75	6888	1056.6	706	1127.2
1.30	677.22	5049	727.71	5183	779.54	5313	832.67	5436	887.03	5555	942.58
1.40	577.70	4052	618.22	4147	659.69	4236	702.05	4323	745.28	4405	789.33
1.50	492.90	3274	525.64	3340	559.04	3404	593.08	3465	627.73	3524	662.97
1.60	421.24	2663	447.87	2711	474.98	2757	502.55	2801	530.56	2842	558.98
1.70	360.92	2182	382.74	2217	404.91	2250	427.41	2283	450.24	2312	473.36
1.80	310.20	1800	328.20	1827	346.47	1851	364.98	1875	383.73	1897	402.70
1.90	267.52	1496	282.48	1516	297.64	1534	312.98	1552	328.50	1568	344.18
2.00	231.55	1252	244.07	1266	256.73	1281	269.54	1293	282.47	1307	295.54
2.10	201.16	1053	211.69	1066	222.35	1075	233.10	1086	243.96	1096	254.92
2.20	175.39	893	184.32	901	193.33	910	202.43	918	211.61	925	220.86
2.30	153.49	760	161.09	768	168.77	773	176.50	780	184.30	786	192.16
2.40	134.79	652	141.31	657	147.88	662	154.50	666	161.16	672	167.88
2.50	118.78	561	124.39	566	130.05	569	135.74	573	141.47	577	147.24
2.60	105.03	485	109.88	489	114.77	492	119.69	495	124.64	498	129.62
2.70	93.160	4220	97.380	4247	101.63	427	105.90	430	110.20	432	114.52
2.80	82.891	3684	86.575	3706	90.281	3729	94.010	3748	97.758	3766	101.52
2.90	73.973	3230	77.203	3248	80.451	3266	83.717	3282	86.999	3298	90.297
3.00	66.204	2842	69.046	2858	71.904	2873	74.777	2885	77.662	2899	80.561

TABLE III—*Continued*

λ (microns)	4500°K		4600°K		4700°K		4800°K		4900°K		5000°K
	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ
0.26	150.49	4588	196.37	5697	253.34	7006	323.40	8533	408.73	10303	511.76
0.27	196.08	5726	253.34	7044	323.78	8581	409.59	10361	513.20	12404	637.24
0.28	249.04	6980	318.84	8509	403.93	10279	506.72	12308	629.80	14620	776.00
0.29	309.23	8330	392.53	10072	493.25	12070	613.95	14343	757.38	16913	926.51
0.30	376.28	9760	473.88	11707	590.95	13926	730.21	16431	894.52	19243	1087.0
0.31	449.70	11244	562.14	13391	696.05	15817	854.22	18539	1039.6	2157	1255.3
0.32	528.81	12763	656.44	15097	807.41	17716	984.57	20636	1190.9	2387	1429.6
0.33	612.86	14295	755.81	16801	923.82	19596	1119.8	2269	1346.7	2610	1607.7
0.34	701.01	15819	859.20	18482	1044.0	2143	1258.3	2469	1505.2	2823	1787.5
0.35	792.36	17319	965.55	20119	1166.7	2321	1398.8	2658	1664.6	3027	1967.3
0.36	886.04	18775	1073.8	2169	1290.7	2490	1539.7	2838	1823.5	3215	2145.0
0.37	981.13	20176	1182.9	2319	1414.8	2649	1679.7	3006	1980.3	3390	2319.3
0.38	1076.8	2151	1291.9	2460	1537.9	2797	1817.6	3160	2133.6	3550	2488.6
0.39	1172.2	2276	1399.8	2592	1659.0	2934	1952.4	3300	2282.4	3693	2651.7
0.40	1266.6	2393	1505.9	2713	1777.2	3058	2083.0	3427	2425.7	3819	2807.6
0.41	1359.3	2500	1609.3	2824	1891.7	3169	2208.6	3539	2562.5	3930	2955.5
0.42	1449.8	2597	1709.5	2922	2001.7	3269	2328.6	3636	2692.2	4024	3094.6
0.43	1537.3	2686	1805.9	3009	2106.8	3355	2442.3	3719	2814.2	4103	3224.5
0.44	1621.6	2763	1897.9	3086	2206.5	3429	2549.4	3788	2928.2	4166	3344.8
0.45	1702.1	2832	1985.3	3152	2300.5	3489	2649.4	3844	3033.8	4216	3455.4
0.46	1778.7	2890	2067.7	3206	2388.3	3539	2742.2	3888	3131.0	4250	3556.0
0.47	1850.9	2939	2144.8	3251	2469.9	3578	2827.7	3918	3219.5	4273	3646.8
0.48	1918.7	2979	2216.6	3285	2545.1	3606	2905.7	3939	3299.6	4283	3727.9
0.49	1981.9	3010	2282.9	3311	2614.0	3623	2976.3	3948	3371.1	4284	3799.5
0.50	2040.4	3033	2343.7	3327	2676.4	3632	3039.6	3948	3434.4	4273	3861.7
0.51	2094.1	3049	2399.0	3336	2732.6	3632	3095.8	3938	3489.6	4253	3914.9
0.52	2143.2	3057	2448.9	3336	2782.5	3624	3144.9	3921	3537.0	4225	3959.5
0.53	2187.5	3059	2493.4	3330	2826.4	3609	3187.3	3895	3576.8	4190	3995.8
0.54	2227.3	3054	2532.7	3317	2864.4	3587	3223.1	3864	3609.5	4147	4024.2
0.55	2262.6	3043	2566.9	3298	2896.7	3559	3252.6	3827	3635.3	4098	4045.1
0.56	2293.5	3027	2596.2	3274	2923.6	3526	3276.2	3783	3654.5	4045	4059.0
0.57	2320.1	3007	2620.8	3245	2945.3	3488	3294.1	3735	3667.6	3987	4066.3
0.58	2342.7	2982	2640.9	3211	2962.0	3446	3306.6	3684	3675.0	3924	4067.4
0.59	2361.3	2953	2656.6	3174	2974.0	3400	3314.0	3628	3676.8	3859	4062.7
0.60	2376.2	2921	2668.3	3133	2981.6	3351	3316.7	3569	3673.6	3791	4052.7
0.61	2387.6	2884	2676.0	3091	2985.1	3298	3314.9	3508	3665.7	3721	4037.8
0.62	2395.5	2846	2680.1	3044	2984.5	3244	3308.9	3445	3653.4	3648	4018.2
0.63	2400.2	2806	2680.8	2996	2980.4	3187	3299.1	3380	3637.1	3574	3994.5
0.64	2401.9	2763	2678.2	2946	2972.8	3129	3285.7	3314	3617.1	3499	3967.0

TABLE III—*Continued*

λ (microns)	4500°K		4600°K		4700°K		4800°K		4900°K		5000°K
	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ
0.65	2400.8	2719	2672.7	2893	2962.0	3070	3269.0	3246	3593.6	3423	3935.9
0.66	2397.0	2673	2664.3	2840	2948.3	3010	3249.3	3178	3567.1	3346	3901.7
0.67	2390.7	2626	2653.3	2787	2932.0	2947	3226.7	3110	3537.7	3270	3864.7
0.68	2382.2	2577	2639.9	2732	2913.1	2886	3201.7	3040	3505.7	3194	3825.1
0.69	2371.4	2529	2624.3	2676	2891.9	2824	3174.3	2971	3471.4	3118	3783.2
0.70	2358.7	2479	2606.6	2621	2868.7	2762	3144.9	2902	3435.1	3042	3739.3
0.71	2344.2	2429	2587.1	2565	2843.6	2699	3113.5	2834	3396.9	2968	3693.7
0.72	2327.9	2380	2565.9	2508	2816.7	2638	3080.5	2766	3357.1	2894	3646.5
0.73	2310.2	2329	2543.1	2453	2788.4	2576	3046.0	2699	3315.9	2820	3597.9
0.74	2291.0	2279	2518.9	2398	2758.7	2515	3010.2	2632	3273.4	2748	3548.2
0.75	2270.6	2229	2493.5	2342	2727.7	2455	2973.2	2567	3229.9	2676	3497.5
0.80	2153.2	1985	2351.7	2076	2559.3	2166	2775.9	2253	3001.2	2340	3235.2
0.85	2019.5	1758	2195.3	1829	2378.2	1901	2568.3	1971	2765.4	2038	2969.2
0.90	1878.9	1550	2033.9	1607	2194.6	1665	2361.1	1719	2533.0	1773	2710.3
0.95	1737.7	1365	1874.2	1410	2015.2	1455	2160.7	1500	2310.7	1542	2464.9
1.00	1600.4	1200	1720.4	1237	1844.1	1273	1971.4	1308	2102.2	1341	2236.3
1.10	1346.7	929	1439.6	953	1534.9	976	1632.5	1000	1732.5	1021	1834.6
1.20	1127.2	723	1199.5	739	1273.4	755	1348.9	769	1425.8	784	1504.2
1.30	942.58	5669	999.27	5778	1057.0	589	1115.9	598	1175.7	608	1236.5
1.40	789.33	4484	834.17	4558	879.75	4630	926.05	4699	973.04	4765	1020.7
1.50	662.97	3578	698.75	3631	735.06	3681	771.87	3728	809.15	3775	846.90
1.60	558.98	2882	587.80	2919	616.99	2955	646.54	2989	676.43	3021	706.64
1.70	473.36	2341	496.77	2369	520.46	2394	544.40	2418	568.58	2442	593.00
1.80	402.70	1918	421.88	1938	441.26	1957	460.83	1975	480.58	1991	500.49
1.90	344.18	1584	360.02	1599	376.01	1613	392.14	1626	408.40	1639	424.79
2.00	295.54	1318	308.72	1329	322.01	1340	335.41	1350	348.91	1359	362.50
2.10	254.92	1105	265.97	1113	277.10	1121	288.31	1129	299.60	1136	310.96
2.20	220.86	932	230.18	938	239.56	945	249.01	950	258.51	956	268.07
2.30	192.16	791	200.07	796	208.03	802	216.05	805	224.10	810	232.20
2.40	167.88	675	174.63	680	181.43	683	188.26	687	195.13	691	202.04
2.50	147.24	580	153.04	583	158.87	587	164.74	589	170.63	592	176.55
2.60	129.62	501	134.63	503	139.66	506	144.72	508	149.80	510	154.90
2.70	114.52	434	118.86	437	123.23	438	127.61	440	132.01	442	136.43
2.80	101.52	379	105.31	380	109.11	382	112.93	383	116.76	384	120.60
2.90	90.297	3312	93.609	3326	96.935	3338	100.27	335	103.62	337	106.99
3.00	80.561	2910	83.471	2922	86.393	2932	89.325	2942	92.267	2951	95.218

TABLE III—*Continued*

λ (microns)	5000°K		5100°K		5200°K		5300°K		5400°K		5500°K
	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ
0.26	511.76	12339	635.15	14660	781.75	17294	954.69	20260	1157.3	2358	1393.1
0.27	6372.4	14733	784.57	17371	958.28	20336	1161.6	2366	1398.2	2733	1671.5
0.28	7760.0	17235	948.35	20172	1150.1	2345	1384.6	2709	1655.5	3111	1966.6
0.29	926.51	19797	1124.5	2301	1354.6	2659	1620.5	3052	1925.7	3483	2274.0
0.30	1087.0	2237	1310.7	2585	1569.2	2968	1866.0	3387	2204.7	3845	2589.2
0.31	1255.3	2494	1504.7	2863	1791.0	3269	2117.9	3710	2488.9	4190	2907.9
0.32	1429.6	2743	1703.9	3133	2017.2	3557	2372.9	4017	2774.6	4513	3225.9
0.33	1607.7	2983	1906.0	3389	2244.9	3829	2627.8	4304	3058.2	4813	3539.5
0.34	1787.5	3212	2108.7	3630	2471.7	4083	2880.0	4568	3336.8	5086	3845.4
0.35	1967.3	3424	2309.7	3855	2695.2	4315	3126.7	4808	3607.5	5331	4140.6
0.36	2145.0	3623	2507.3	4059	2913.2	4526	3365.8	5022	3868.0	5548	4422.8
0.37	2319.3	3803	2699.6	4244	3124.0	4714	3595.4	5210	4116.4	5734	4689.8
0.38	2488.6	3966	2885.2	4408	3326.0	4878	3813.8	5372	4351.0	5892	4940.2
0.39	2651.7	4110	3062.7	4552	3517.9	5018	4019.7	5509	4570.6	6022	5172.8
0.40	2807.6	4236	3231.2	4675	3698.7	5136	4212.3	5620	4774.3	6125	5386.8
0.41	2955.5	4343	3389.8	4777	3867.5	5233	4390.8	5708	4961.6	6202	5581.8
0.42	3094.6	4432	3537.8	4861	4023.9	5308	4554.7	5773	5132.0	6255	5757.5
0.43	3224.5	4505	3675.0	4925	4167.5	5363	4703.8	5817	5285.5	6286	5914.1
0.44	3344.8	4561	3800.9	4972	4298.1	5400	4838.1	5840	5422.1	6297	6051.8
0.45	3455.4	4601	3915.5	5003	4415.8	5418	4957.6	5847	5542.3	6287	6171.0
0.46	3556.0	4628	4018.8	5018	4520.6	5421	5062.7	5836	5646.3	6261	6272.4
0.47	3646.8	4641	4110.9	5019	4612.8	5409	5153.7	5810	5734.7	6220	6356.7
0.48	3727.9	4640	4191.9	5008	4692.7	5384	5231.1	5770	5808.1	6165	6424.6
0.49	3799.5	4628	4262.3	4984	4760.7	5347	5295.4	5719	5867.3	6096	6476.9
0.50	3861.7	4607	4322.4	4949	4817.3	5299	5347.2	5656	5912.8	6019	6514.7
0.51	3914.9	4576	4372.5	4905	4863.0	5242	5387.2	5584	5945.6	5931	6538.7
0.52	3959.5	4536	4413.1	4853	4898.4	5176	5416.0	5504	5966.4	5836	6550.0
0.53	3995.8	4488	4444.6	4794	4924.0	5103	5434.3	5416	5975.9	5734	6549.3
0.54	4024.2	4434	4467.6	4728	4940.4	5023	5442.7	5324	5975.1	5625	6537.6
0.55	4045.1	4375	4482.6	4655	4948.1	4939	5442.0	5225	5964.5	5514	6515.9
0.56	4059.0	4310	4490.0	4579	4947.9	4849	5432.8	5123	5945.1	5397	6484.8
0.57	4066.3	4241	4490.4	4498	4940.2	4756	5415.8	5017	5917.5	5278	6445.3
0.58	4067.4	4168	4484.2	4414	4925.6	4660	5391.6	4909	5882.5	5157	6398.2
0.59	4062.7	4093	4472.0	4326	4904.6	4562	5360.8	4798	5840.6	5035	6344.1
0.60	4052.7	4014	4454.1	4237	4877.8	4462	5324.0	4687	5792.7	4911	6283.8
0.61	4037.8	3933	4431.1	4146	4845.7	4361	5281.8	4574	5739.2	4787	6217.9
0.62	4018.2	3851	4403.3	4055	4808.8	4258	5234.6	4462	5680.8	4664	6147.2
0.63	3994.5	3768	4371.3	3962	4767.5	4155	5183.0	4349	5617.9	4542	6072.1
0.64	3967.0	3683	4335.3	3869	4722.2	4053	5127.5	4237	5551.2	4419	5993.1

TABLE III—*Continued*

λ (microns)	5000°K		5100°K		5200°K		5300°K		5400°K		5500°K
	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ
0.65	3935.9	3599	4295.9	3775	4673.4	3952	5068.6	4125	5481.1	4299	5911.0
0.66	3901.7	3515	4258.2	3683	4621.5	3850	5006.5	4015	5408.0	4180	5826.0
0.67	3864.7	3431	4207.8	3590	4566.8	3749	4941.7	3907	5332.4	4062	5738.6
0.68	3825.1	3347	4159.8	3499	4509.7	3649	4874.6	3799	5254.5	3947	5649.2
0.69	3783.2	3264	4109.6	3408	4450.4	3552	4805.6	3693	5174.9	3833	5558.2
0.70	3739.3	3182	4057.5	3318	4389.3	3455	4734.8	3590	5093.8	3722	5466.0
0.71	3693.7	3099	4003.6	3231	4326.7	3359	4662.6	3488	5011.4	3614	5372.8
0.72	3646.5	3019	3948.4	3143	4262.7	3267	4589.4	3388	4928.2	3507	5278.9
0.73	3597.9	2940	3891.9	3058	4197.7	3175	4515.2	3290	4844.2	3404	5184.6
0.74	3548.2	2862	3834.4	2974	4131.8	3086	4440.4	3195	4759.9	3302	5090.1
0.75	3497.5	2785	3776.0	2893	4065.3	2998	4365.1	3102	4675.3	3203	4995.6
0.80	3235.2	2425	3477.7	2509	3728.6	2590	3987.6	2671	4254.7	2749	4529.6
0.85	2969.2	2106	3179.8	2170	3396.8	2235	3620.3	2296	3849.9	2357	4085.6
0.90	2710.3	1826	2892.9	1877	3080.6	1926	3273.2	1975	3470.7	2022	3672.9
0.95	2464.9	1583	2623.2	1624	2785.6	1662	2951.8	1701	3121.9	1738	3295.7
1.00	2236.3	1375	2373.8	1407	2514.5	1437	2658.2	1468	2805.0	1496	2954.6
1.10	1834.6	1042	1938.8	1062	2045.0	1082	2153.2	1101	2263.3	1119	2375.2
1.20	1504.2	798	1584.0	810	1665.0	824	1747.4	836	1831.0	847	1915.7
1.30	1236.5	617	1298.2	626	1360.8	634	1424.2	642	1488.4	650	1553.4
1.40	1020.7	483	1069.0	488	1117.8	495	1167.3	500	1217.3	505	1267.8
1.50	846.90	3818	885.08	3860	923.68	3899	962.67	3937	1002.0	398	1041.8
1.60	706.64	3052	737.16	3081	767.97	3109	799.06	3137	830.43	3161	862.04
1.70	593.00	2464	617.64	2485	642.49	2505	667.54	2525	692.79	2543	718.22
1.80	500.49	2008	520.57	2024	540.81	2038	561.19	2052	581.71	2066	602.37
1.90	424.79	1651	441.30	1662	457.92	1674	474.66	1683	491.49	1693	508.42
2.00	362.50	1368	376.18	1377	389.95	1385	403.80	1393	417.73	1401	431.74
2.10	310.96	1143	322.39	1149	333.88	1156	345.44	1161	357.05	1167	368.72
2.20	268.07	962	277.69	966	287.35	971	297.06	976	306.82	980	316.62
2.30	232.20	814	240.34	818	248.52	822	256.74	825	264.99	828	273.27
2.40	202.04	693	208.97	697	215.94	700	222.94	702	229.96	705	237.01
2.50	176.55	594	182.49	597	188.46	599	194.45	601	200.46	603	206.49
2.60	154.90	512	160.02	514	165.16	516	170.32	518	175.50	519	180.69
2.70	136.43	444	140.87	445	145.32	446	149.78	448	154.26	449	158.75
2.80	120.60	386	124.46	387	128.33	389	132.22	389	136.11	391	140.02
2.90	106.99	337	110.36	338	113.74	339	117.13	341	120.54	341	123.95
3.00	95.218	2960	98.178	2969	101.15	297	104.12	299	107.11	299	110.10

TABLE III—*Continued*

λ (microns)	5500°K		5600°K		5700°K		5800°K		5900°K		6000°K
	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ
0.26	1393.1	2728	1665.9	3138	1979.7	3588	2338.5	4085	2747.0	4624	3209.4
0.27	1671.5	3142	1985.7	3589	2344.6	4080	2752.6	4616	3214.2	5195	3733.7
0.28	1966.6	3553	2321.9	4035	2725.4	4561	3181.5	5129	3694.4	5744	4268.8
0.29	2274.0	3955	2669.5	4467	3116.2	5021	3618.3	5618	4180.1	6259	4806.0
0.30	2589.2	4341	3023.3	4878	3511.1	5455	4056.6	6074	4664.0	6735	5337.5
0.31	2907.9	4706	3378.5	5262	3904.7	5858	4490.5	6493	5139.8	7167	5856.5
0.32	3225.9	5046	3730.5	5617	4292.2	6224	4914.6	6869	5601.5	7553	6356.8
0.33	3539.5	5358	4075.3	5937	4669.0	6552	5324.2	7203	6044.5	7889	6833.4
0.34	3845.4	5638	4409.2	6223	5031.5	6841	5715.6	7492	6464.8	8175	7282.3
0.35	4140.6	5886	4729.2	6473	5376.5	7089	6085.4	7736	6859.0	8413	7700.3
0.36	4422.8	6102	5033.0	6685	5701.5	7298	6431.3	7937	7225.0	8604	8085.4
0.37	4689.8	6286	5318.4	6863	6004.7	7467	6751.4	8096	7561.0	8749	8435.9
0.38	4940.2	6438	5584.0	7006	6284.6	7600	7044.6	8215	7866.1	8852	8751.3
0.39	5172.8	6559	5828.7	7117	6540.4	7697	7310.1	8296	8139.7	8916	9031.3
0.40	5386.8	6651	6051.9	7197	6771.6	7760	7547.6	8344	8382.0	8943	9276.3
0.41	5581.8	6716	6253.4	7246	6978.0	7795	7757.5	8358	8593.3	8938	9487.1
0.42	5757.5	6755	6433.0	7270	7160.0	7801	7940.1	8344	8774.5	8902	9664.7
0.43	5914.1	6771	6591.2	7269	7318.1	7781	8096.2	8304	8926.6	8840	9810.6
0.44	6051.8	6765	6728.3	7246	7452.9	7738	8226.7	8242	9050.9	8754	9926.3
0.45	6171.0	6740	6845.0	7203	7565.3	7676	8332.9	8157	9148.6	8648	10013.
0.46	6272.4	6697	6942.1	7142	7656.3	7595	8415.8	8056	9221.4	8524	10074.
0.47	6356.7	6638	7020.5	7065	7727.0	7499	8476.9	7939	9270.8	8385	10109.
0.48	6424.6	6566	7081.2	6974	7778.6	7389	8517.5	7809	9298.4	8233	10122.
0.49	6476.9	6482	7125.1	6872	7812.3	7268	8539.1	7667	9305.8	8070	10113.
0.50	6514.7	6387	7153.4	6759	7829.3	7137	8543.0	7516	9294.6	7899	10084.
0.51	6538.7	6283	7167.0	6638	7830.8	6997	8530.5	7358	9266.3	7721	10038.
0.52	6550.0	6171	7167.1	6510	7818.1	6850	8503.1	7194	9222.5	7537	9976.2
0.53	6549.3	6053	7154.6	6376	7792.2	6699	8462.1	7024	9164.5	7351	9899.6
0.54	6537.6	5931	7130.7	6236	7754.3	6544	8408.7	6852	9093.9	7160	9809.9
0.55	6515.9	5803	7096.2	6094	7705.6	6385	8344.1	6678	9011.9	6969	9708.8
0.56	6484.8	5673	7052.1	5949	7647.0	6225	8269.5	6502	8919.7	6777	9597.4
0.57	6445.3	5540	6999.3	5802	7579.5	6064	8185.9	6326	8818.5	6586	9477.1
0.58	6398.2	5405	6938.7	5655	7504.2	5902	8094.4	6150	8709.4	6395	9348.9
0.59	6344.1	5270	6871.1	5507	7421.8	5741	7995.9	5975	8593.4	6207	9214.1
0.60	6283.8	5135	6797.3	5359	7333.2	5580	7891.2	5802	8471.4	6021	9073.5
9.61	6217.9	5000	6717.9	5212	7239.1	5422	7781.3	5631	8344.4	5838	8928.2
0.62	6147.2	4865	6633.7	5066	7140.3	5265	7666.8	5463	8213.1	5657	8778.8
0.63	6072.1	4732	6545.3	4922	7037.5	5110	7548.5	5297	8078.2	5481	8626.3
0.64	5993.1	4601	6453.2	4780	6931.2	4958	7427.0	5134	7940.4	5309	8471.3

TABLE III—*Continued*

λ (microns)	5500°K		5600°K		5700°K		5800°K		5900°K		6000°K
	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ
0.65	5911.0	4470	6358.0	4640	6822.0	4809	7302.9	4975	7800.4	5140	8314.4
0.66	5826.0	4342	6260.2	4503	6710.5	4663	7176.8	4819	7658.7	4975	8156.2
0.67	5738.6	4216	6160.2	4369	6597.1	4519	7049.0	4668	7515.8	4814	7997.2
0.68	5649.2	4093	6058.5	4237	6482.2	4380	6920.2	4519	7372.1	4658	7837.9
0.69	5558.2	3972	5955.4	4109	6366.3	4243	6790.6	4375	7228.1	4507	7678.8
0.70	5466.0	3854	5851.4	3982	6249.6	4111	6660.7	4235	7084.2	4359	7520.1
0.71	5372.8	3738	5746.6	3860	6132.6	3981	6530.7	4099	6940.6	4216	7362.2
0.72	5278.9	3625	5641.4	3741	6015.5	3855	6401.0	3967	6797.7	4077	7205.4
0.73	5184.6	3515	5536.1	3625	5898.6	3733	6271.9	3839	6655.8	3942	7050.0
0.74	5090.1	3408	5430.9	3513	5782.2	3614	6143.6	3714	6515.0	3812	6896.2
0.75	4995.6	3304	5326.0	3403	5666.3	3499	6016.2	3593	6375.5	3686	6744.1
0.80	4529.6	2826	4812.2	2902	5102.4	2974	5399.8	3047	5704.5	3116	6016.1
0.85	4085.6	2416	4327.2	2474	4574.6	2531	4827.7	2584	5086.1	2639	5350.0
0.90	3672.9	2068	3879.7	2113	4091.0	2155	4306.5	2198	4526.3	2239	4750.2
0.95	3295.7	1773	3473.0	1808	3653.8	1841	3837.9	1874	4025.3	1906	4215.9
1.00	2954.6	1524	3107.0	1551	3262.1	1578	3419.9	1603	3580.2	1628	3743.0
1.10	2375.2	1136	2488.8	1154	2604.2	1170	2721.2	1186	2839.8	1200	2959.8
1.20	1915.7	859	2001.6	869	2088.5	880	2176.5	891	2265.6	899	2355.5
1.30	1553.4	658	1619.2	665	1685.7	671	1752.8	679	1820.7	684	1889.1
1.40	1267.8	510	1318.8	515	1370.3	520	1422.3	524	1474.7	528	1527.5
1.50	1041.8	401	1081.9	404	1122.3	407	1163.0	411	1204.1	413	1245.4
1.60	862.04	3186	893.90	3210	926.00	3232	958.32	3253	990.85	3274	1023.6
1.70	718.22	2560	743.82	2577	769.59	2593	795.52	2608	821.60	2622	847.82
1.80	602.37	2078	623.15	2090	644.05	2102	665.07	2113	686.20	2124	707.44
1.90	508.42	1703	525.45	1712	542.57	1720	559.77	1728	577.05	1736	594.41
2.00	431.74	1407	445.81	1414	459.95	1420	474.15	1426	488.41	1432	502.73
2.10	368.72	1173	380.45	1177	392.22	1182	404.04	1187	415.91	1191	427.82
2.20	316.62	984	326.46	988	336.34	991	346.25	995	356.20	999	366.19
2.30	273.27	832	281.59	835	289.94	837	298.31	840	306.71	843	315.14
2.40	237.01	707	244.08	710	251.18	712	258.30	714	265.44	716	272.60
2.50	206.49	606	212.55	607	218.62	609	224.71	610	230.81	612	236.93
2.60	180.69	521	185.90	522	191.12	523	196.35	525	201.60	527	206.87
2.70	158.75	451	163.26	451	167.77	453	172.30	454	176.84	455	181.39
2.80	140.02	391	143.93	393	147.86	393	151.79	394	155.73	396	159.69
2.90	123.95	341	127.36	343	130.79	343	134.22	345	137.67	344	141.11
3.00	110.10	300	113.10	300	116.10	301	119.11	302	122.13	302	125.15

TABLE III—*Continued*

λ (microns)	6000°K		6100°K		6200°K		6300°K		6400°K		6500°K
	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ
0.26	3209.4	5213	3730.7	5849	4315.6	6536	4969.2	7275	5696.7	8066	6503.3
0.27	3733.7	5824	4316.1	6499	4966.0	7223	5688.3	7999	6488.2	8825	7370.7
0.28	4268.8	6403	4909.1	7110	5620.1	7864	6406.5	8667	7273.2	9519	8225.1
0.29	4806.0	6944	5500.4	7674	6267.8	8449	7112.7	9271	8039.8	10139	9053.7
0.30	5337.5	7439	6081.4	8184	6899.8	8974	7797.2	9805	8777.7	10679	9845.6
0.31	5856.5	7882	6644.7	8638	7508.5	9432	8451.7	10267	9478.4	11141	10592.
0.32	6356.8	8273	7184.1	9030	8087.1	9825	9069.6	10656	10135.	1152	11287.
0.33	6833.4	8609	7694.3	9363	8630.6	10151	9645.7	10973	10743.	1183	11926.
0.34	7282.3	8890	8171.3	9637	9135.0	10413	10176.	1122	11298.	1206	12504.
0.35	7700.3	9119	8612.2	9853	9597.5	10615	10659.	1140	11799.	1222	13021.
0.36	8085.4	9296	9015.0	10015	10016.	1076	11092.	1153	12245.	1231	13476.
0.37	8435.9	9426	9378.5	10125	10391.	1085	11476.	1159	12635.	1235	13870.
0.38	8751.3	9510	9702.3	10189	10721.	1089	11810.	1160	12970.	1234	14204.
0.39	9031.3	9553	9986.6	10210	11008.	1088	12096.	1157	13253.	1227	14480.
0.40	9276.3	9559	10232.	1019	11251.	1084	12335.	1149	13484.	1217	14701.
0.41	9487.1	9530	10440.	1014	11454.	1075	12529.	1139	13668.	1203	14871.
0.42	9664.7	9471	10612.	1005	11617.	1064	12681.	1125	13806.	1185	14991.
0.43	9810.6	9385	10749.	994	11743.	1051	12794.	1107	13901.	1166	15067.
0.44	9926.3	9275	10854.	980	11834.	1035	12869.	1088	13957.	1144	15101.
0.45	10013.	915	10928.	965	11893.	1016	12909.	1067	13976.	1120	15096.
0.46	10074.	900	10974.	947	11921.	996	12917.	1045	13962.	1094	15056.
0.47	10109.	884	10993.	929	11922.	975	12897.	1021	13918.	1067	14985.
0.48	10122.	866	10988.	909	11897.	953	12850.	996	13846.	1040	14886.
0.49	10113.	847	10960.	889	11849.	930	12779.	970	13749.	1012	14761.
0.50	10084.	829	10913.	867	11780.	906	12686.	944	13630.	984	14614.
0.51	10038.	809	10847.	845	11692.	882	12574.	918	13492.	955	14447.
0.52	9976.2	7882	10764.	823	11587.	857	12444.	892	13336.	926	14262.
0.53	9899.6	7676	10667.	800	11467.	833	12300.	865	13165.	898	14063.
0.54	9809.9	7469	10557.	777	11334.	809	12143.	838	12981.	870	13851.
0.55	9708.8	7259	10435.	755	11190.	784	11974.	812	12786.	841	13627.
0.56	9597.4	7052	10303.	732	11035.	760	11795.	787	12582.	813	13395.
0.57	9477.1	6845	10162.	710	10872.	736	11608.	761	12369.	786	13155.
0.58	9348.9	6641	10013.	688	10701.	713	11414.	736	12150.	760	12910.
0.59	9214.1	6438	9857.9	6667	10525.	689	11214.	712	11926.	734	12660.
0.60	9073.5	6239	9697.4	6454	10343.	667	11010.	687	11697.	709	12406.
0.61	8928.2	6043	9532.5	6246	10157.	645	10802.	664	11466.	684	12150.
0.62	8778.8	5852	9364.0	6042	9968.2	6232	10591.	642	11233.	660	11893.
0.63	8626.3	5664	9192.7	5843	9777.0	6022	10379.	620	10999.	637	11636.
0.64	8471.3	5480	9019.3	5650	9584.3	5818	10166.	598	10764.	615	11379.

TABLE III—*Continued*

λ (microns)	6000°K		6100°K		6200°K		6300°K		6400°K		6500°K
	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ
0.65	8314.4	5301	8844.5	5462	9390.7	5619	9952.6	5774	10530.	593	11123.
0.66	8156.2	5127	8668.9	5279	9196.8	5426	9739.4	5572	10297.	571	10868.
0.67	7997.2	4958	8493.0	5101	9003.1	5239	9527.0	5378	10065.	551	10616.
0.68	7837.9	4794	8317.3	4928	8810.1	5058	9315.9	5189	9834.8	5315	10366.
0.69	7678.8	4634	8142.2	4760	8618.2	4884	9106.6	5006	9607.2	5125	10120.
0.70	7520.1	4479	7968.0	4598	8427.8	4715	8899.3	4830	9382.3	4941	9876.4
0.71	7362.2	4329	7795.1	4442	8239.3	4551	8694.4	4659	9160.3	4765	9636.8
0.72	7205.4	4184	7623.8	4290	8052.8	4394	8492.2	4495	8941.7	4594	9401.1
0.73	7050.0	4044	7454.4	4143	7868.7	4241	8292.8	4337	8726.5	4430	9169.5
0.74	6896.2	3908	7287.0	4002	7687.2	4094	8096.6	4184	8515.0	4272	8942.2
0.75	6744.1	3777	7121.8	3866	7508.4	3952	7903.6	4037	8307.3	4120	8719.3
0.80	6016.1	3185	6334.6	3251	6659.7	3317	6991.4	3380	7329.4	3442	7673.6
0.85	5350.0	2690	5619.0	2740	5893.0	2790	6172.0	2838	6455.8	2884	6744.2
0.90	4750.2	2278	4978.0	2317	5209.7	2355	5445.2	2391	5684.3	2426	5926.9
0.95	4215.9	1936	4409.5	1965	4606.0	1995	4805.5	2022	5007.7	2050	5212.7
1.00	3743.0	1651	3908.1	1674	4075.5	1697	4245.2	1718	4417.0	1739	4590.9
1.10	2959.8	1216	3081.4	1230	3204.4	1243	3328.7	1257	3454.4	1269	3581.3
1.20	2355.5	910	2446.5	918	2538.3	927	2631.0	935	2724.5	943	2818.8
1.30	1889.1	691	1958.2	697	2027.9	702	2098.1	707	2168.8	713	2240.1
1.40	1527.5	533	1580.8	536	1634.4	540	1688.4	544	1742.8	547	1797.5
1.50	1245.4	416	1287.0	419	1328.9	421	1371.0	424	1413.4	426	1456.0
1.60	1023.6	329	1056.5	331	1089.6	333	1122.9	335	1156.4	336	1190.0
1.70	847.82	2637	874.19	2650	900.69	2663	927.32	2675	954.07	2687	980.94
1.80	707.44	2133	728.77	2143	750.20	2153	771.73	2161	793.34	2170	815.04
1.90	594.41	1744	611.85	1751	629.36	1757	646.93	1764	664.57	1770	682.27
2.00	502.73	1438	517.11	1443	531.54	1448	546.02	1453	560.55	1458	575.13
2.10	427.82	1196	439.78	1199	451.77	1203	463.80	1207	475.87	1211	487.98
2.20	366.19	1001	376.20	1005	386.25	1008	396.33	1010	406.43	1013	416.56
2.30	315.14	845	323.59	848	332.07	850	340.57	852	349.09	854	357.63
2.40	272.60	718	279.78	720	286.98	721	294.19	724	301.43	725	308.68
2.50	236.93	614	243.07	615	249.22	617	255.39	618	261.57	619	267.76
2.60	206.87	527	212.14	529	217.43	530	222.73	531	228.04	531	233.35
2.70	181.39	456	185.95	456	190.51	458	195.09	459	199.68	459	204.27
2.80	159.69	395	163.64	397	167.61	397	171.58	398	175.56	399	179.55
2.90	141.11	346	144.57	346	148.03	346	151.49	348	154.97	347	158.44
3.00	125.15	303	128.18	303	131.21	303	134.24	305	137.29	304	140.33

TABLE III—*Continued*

λ (microns)	6500°K		6600°K		6700°K		6800°K		6900°K		7000°K
	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ
0.26	6503.3	8911	7394.4	9811	8375.5	10765	9452.0	11777	10630.	1284	11914.
0.27	7370.7	9703	8341.0	10633	9404.3	11615	10566.	1265	11831.	1374	13205.
0.28	8225.1	10418	9266.9	11367	10404.	1236	11640.	1342	12982.	1451	14433.
0.29	9053.7	11050	10159.	1201	11360.	1301	12661.	1406	14067.	1516	15583.
0.30	9845.6	11596	11005.	1256	12261.	1355	13616.	1460	15076.	1567	16643.
0.31	10592.	1206	11798.	1300	13098.	1399	14497.	1502	15999.	1608	17607.
0.32	11287.	1243	12530.	1336	13866.	1433	15299.	1533	16832.	1637	18469.
0.33	11926.	1271	13197.	1363	14560.	1457	16017.	1555	17572.	1655	19227.
0.34	12504.	1292	13796.	1381	15177.	1473	16650.	1568	18218.	1663	19881.
0.35	13021.	1306	14327.	1392	15719.	1480	17199.	1571	18770.	1664	20434.
0.36	13476.	1312	14788.	1396	16184.	1481	17665.	1568	19233.	1656	20889.
0.37	13870.	1313	15183.	1393	16576.	1475	18051.	1557	19608.	1643	21251.
0.38	14204.	1308	15512.	1385	16897.	1463	18360.	1542	19902.	1622	21524.
0.39	14480.	1299	15779.	1372	17151.	1446	18597.	1521	20118.	1597	21715.
0.40	14701.	1285	15986.	1355	17341.	1425	18766.	1496	20262.	1568	21830.
0.41	14871.	1267	16138.	1334	17472.	1400	18872.	1468	20340.	1536	21876.
0.42	14991.	1248	16239.	1309	17548.	1373	18921.	1436	20357.	1501	21858.
0.43	15067.	1224	16291.	1283	17574.	1343	18917.	1402	20319.	1463	21782.
0.44	15101.	1198	16299.	1255	17554.	1311	18865.	1367	20232.	1424	21656.
0.45	15096.	1172	16268.	1224	17492.	1277	18769.	1331	20100.	1383	21483.
0.46	15056.	1144	16200.	1193	17393.	1243	18636.	1292	19928.	1343	21271.
0.47	14985.	1114	16099.	1161	17260.	1207	18467.	1255	19722.	1301	21023.
0.48	14886.	1084	15970.	1127	17097.	1172	18269.	1216	19485.	1259	20744.
0.49	14761.	1053	15814.	1095	16909.	1135	18044.	1177	19221.	1218	20439.
0.50	14614.	1022	15636.	1061	16697.	1100	17797.	1138	18935.	1176	20111.
0.51	14447.	991	15438.	1028	16466.	1063	17529.	1100	18629.	1136	19765.
0.52	14262.	961	15223.	994	16217.	1028	17245.	1062	18307.	1096	19403.
0.53	14063.	929	14992.	962	15954.	994	16948.	1025	17973.	1056	19029.
0.54	13851.	899	14750.	929	15679.	959	16638.	989	17627.	1018	18645.
0.55	13627.	870	14497.	897	15394.	926	16320.	953	17273.	981	18254.
0.56	13395.	840	14235.	867	15102.	892	15994.	919	16913.	944	17857.
0.57	13155.	812	13967.	836	14803.	861	15664.	884	16548.	909	17457.
0.58	12910.	783	13693.	807	14500.	829	15329.	852	16181.	875	17056.
0.59	12660.	756	13416.	777	14193.	800	14993.	820	15813.	841	16654.
0.60	12406.	730	13136.	749	13885.	770	14655.	790	15445.	809	16254.
0.61	12150.	704	12854.	723	13577.	741	14318.	760	15078.	778	15856.
0.62	11893.	679	12572.	696	13268.	714	13982.	731	14713.	748	15461.
0.63	11636.	654	12290.	671	12961.	687	13648.	704	14352.	719	15071.
0.64	11379.	630	12009.	646	12655.	662	13317.	677	13994.	692	14686.

TABLE III—*Continued*

λ (microns)	6500°K		6600°K		6700°K		6800°K		6900°K		7000°K
	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ
0.65	11123.	607	11730.	623	12353.	637	12990.	651	13641.	665	14306.
0.66	10868.	586	11454.	599	12053.	614	12667.	626	13293.	640	13933.
0.67	10616.	564	11180.	578	11758.	590	12348.	603	12951.	615	13566.
0.68	10366.	544	10910.	556	11466.	568	12034.	580	12614.	592	13206.
0.69	10120.	524	10644.	535	11179.	547	11726.	558	12284.	569	12853.
0.70	9876.4	5052	10382.	515	10897.	527	11424.	537	11961.	547	12508.
0.71	9636.8	4868	10124.	497	10621.	506	11127.	517	11644.	526	12170.
0.72	9401.1	4692	9870.3	4786	10349.	488	10837.	497	11334.	506	11840.
0.73	9169.5	4522	9621.7	4611	10083.	470	10553.	478	11031.	487	11518.
0.74	8942.2	4358	9378.0	4443	9822.3	4524	10275.	460	10735.	469	11204.
0.75	8719.3	4201	9139.4	4280	9567.4	4358	10003.	444	10447.	450	10897.
0.80	7673.6	3502	8023.8	3560	8379.8	3618	8741.6	3674	9109.0	3728	9481.8
0.85	6744.2	2929	7037.1	2973	7334.4	3016	7636.0	3058	7941.8	3098	8251.6
0.90	5926.9	2460	6172.9	2493	6422.2	2526	6674.8	2557	6930.5	2588	7189.3
0.95	5212.7	2075	5420.2	2101	5630.3	2126	5842.9	2149	6057.8	2173	6275.1
1.00	4590.9	1760	4766.9	1778	4944.7	1798	5124.5	1816	5306.1	1834	5489.5
1.10	3581.3	1282	3709.5	1294	3838.9	1305	3969.4	1316	4101.0	1327	4233.7
1.20	2818.8	951	2913.9	959	3009.8	966	3106.4	972	3203.6	980	3301.6
1.30	2240.1	718	2311.9	723	2384.2	727	2456.9	732	2530.1	737	2603.8
1.40	1797.5	551	1852.6	554	1908.0	557	1963.7	560	2019.7	563	2076.0
1.50	1456.0	429	1498.9	431	1542.0	433	1585.2	435	1628.7	437	1672.4
1.60	1190.0	338	1223.8	340	1257.8	341	1291.9	342	1326.1	344	1360.5
1.70	980.94	2697	1007.9	271	1035.0	272	1062.2	273	1089.5	274	1116.9
1.80	815.04	2168	836.72	2196	858.68	2193	880.61	2201	902.62	2207	924.69
1.90	682.27	1776	700.03	1782	717.85	1788	735.73	1793	753.66	1797	771.63
2.00	575.13	1462	589.75	1466	604.41	1471	619.12	1474	633.86	1478	648.64
2.10	487.98	1214	500.12	1217	512.29	1220	524.49	1223	536.72	1226	548.98
2.20	416.56	1016	426.72	1018	436.90	1021	447.11	1022	457.33	1025	467.58
2.30	357.63	856	366.19	858	374.77	860	383.37	862	391.99	863	400.62
2.40	308.68	726	315.94	728	323.22	729	330.51	731	337.82	732	345.14
2.50	267.76	620	273.96	622	280.18	622	286.40	624	292.64	625	298.89
2.60	233.35	533	238.68	534	244.02	535	249.37	535	254.72	536	260.08
2.70	204.27	460	208.87	461	213.48	461	218.09	463	222.72	462	227.34
2.80	179.55	399	183.54	400	187.54	401	191.55	401	195.56	401	199.57
2.90	158.44	348	161.92	349	165.41	349	168.90	350	172.40	350	175.90
3.00	140.33	305	143.38	306	146.44	305	149.49	306	152.55	307	155.62

TABLE III—*Continued*

λ (microns)	7000°K		7100°K		7200°K		7300°K		7400°K		7500°K
	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ
0.26	11914.	1397	13311.	1515	14826.	1639	16465.	1768	18233.	1903	20136.
0.27	13205.	1488	14693.	1607	16300.	1732	18032.	1861	19893.	1997	21890.
0.28	14433.	1565	15998.	1685	17683.	1808	19491.	1938	21429.	2071	23500.
0.29	15583.	1629	17212.	1747	18959.	1870	20829.	1996	22825.	2127	24952.
0.30	16643.	1680	18323.	1796	20119.	1915	22034.	2039	24073.	2166	26239.
0.31	17607.	1718	19325.	1830	21155.	1947	23102.	2067	25169.	2189	27358.
0.32	18469.	1743	20212.	1853	22065.	1965	24030.	2081	26111.	2198	28309.
0.33	19227.	1758	20985.	1864	22849.	1971	24820.	2082	26902.	2195	29097.
0.34	19881.	1763	21644.	1865	23509.	1967	25476.	2073	27549.	2180	29729.
0.35	20434.	1759	22193.	1856	24049.	1954	26003.	2054	28057.	2156	30213.
0.36	20889.	1747	22636.	1839	24475.	1933	26408.	2027	28435.	2124	30559.
0.37	21251.	1728	22979.	1816	24795.	1904	26699.	1994	28693.	2084	30777.
0.38	21524.	1704	23228.	1787	25015.	1870	26885.	1954	28839.	2040	30879.
0.39	21715.	1675	23390.	1752	25142.	1831	26973.	1911	28884.	1990	30874.
0.40	21830.	1642	23472.	1714	25186.	1789	26975.	1863	28838.	1937	30775.
0.41	21876.	1604	23480.	1674	25154.	1742	26896.	1813	28709.	1883	30592.
0.42	21858.	1565	23423.	1630	25053.	1694	26747.	1761	28508.	1825	30333.
0.43	21782.	1524	23306.	1584	24890.	1646	26536.	1706	28242.	1767	30009.
0.44	21656.	1480	23136.	1538	24674.	1595	26269.	1651	27920.	1709	29629.
0.45	21483.	1437	22920.	1490	24410.	1544	25954.	1596	27550.	1650	29200.
0.46	21271.	1392	22663.	1442	24105.	1492	25597.	1541	27138.	1591	28729.
0.47	21023.	1347	22370.	1394	23764.	1441	25205.	1486	26691.	1533	28224.
0.48	20744.	1303	22047.	1346	23393.	1389	24782.	1433	26215.	1475	27690.
0.49	20439.	1258	21697.	1299	22996.	1339	24335.	1380	25715.	1419	27134.
0.50	20111.	1215	21326.	1252	22578.	1290	23868.	1327	25195.	1364	26559.
0.51	19765.	1171	20936.	1207	22143.	1241	23384.	1276	24660.	1311	25971.
0.52	19403.	1129	20532.	1162	21694.	1194	22888.	1227	24115.	1258	25373.
0.53	19029.	1087	20116.	1118	21234.	1149	22383.	1178	23561.	1209	24770.
0.54	18645.	1047	19692.	1075	20767.	1104	21871.	1132	23003.	1160	24163.
0.55	18254.	1007	19261.	1034	20295.	1061	21356.	1087	22443.	1112	23555.
0.56	17857.	969	18826.	994	19820.	1019	20839.	1043	21882.	1067	22949.
0.57	17457.	932	18389.	956	19345.	978	20323.	1001	21324.	1024	22348.
0.58	17056.	896	17952.	918	18870.	940	19810.	960	20770.	981	21751.
0.59	16654.	862	17516.	882	18398.	902	19300.	921	20221.	941	21162.
0.60	16254.	828	17082.	847	17929.	866	18795.	884	19679.	902	20581.
0.61	15856.	796	16652.	814	17466.	831	18297.	848	19145.	865	20010.
0.62	15461.	765	16226.	782	17008.	798	17806.	813	18619.	830	19449.
0.63	15071.	735	15806.	751	16557.	766	17323.	780	18103.	796	18899.
0.64	14686.	706	15392.	721	16113.	735	16848.	749	17597.	763	18360.

TABLE III—*Continued*

λ (microns)	7000°K		7100°K		7200°K		7300°K		7400°K		7500°K
	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ
0.65	14306.	679	14985.	692	15677.	706	16383.	719	17102.	731	17833.
0.66	13933.	652	14585.	665	15250.	677	15927.	690	16617.	701	17318.
0.67	13566.	627	14193.	638	14831.	651	15482.	661	16143.	673	16816.
0.68	13206.	602	13808.	614	14422.	624	15046.	635	15681.	646	16327.
0.69	12853.	579	13432.	590	14022.	599	14621.	610	15231.	619	15850.
0.70	12508.	556	13064.	567	13631.	576	14207.	585	14792.	594	15386.
0.71	12170.	535	12705.	545	13250.	553	13803.	562	14365.	570	14935.
0.72	11840.	515	12355.	523	12878.	531	13409.	540	13949.	547	14496.
0.73	11518.	495	12013.	503	12516.	510	13026.	519	13545.	525	14070.
0.74	11204.	476	11680.	483	12163.	491	12654.	498	13152.	505	13657.
0.75	10897.	458	11355.	465	11820.	472	12292.	478	12770.	485	13255.
0.80	9481.8	3781	9859.9	3833	10243.	389	10632.	393	11025.	398	11423.
0.85	8251.6	3138	8565.4	3176	8883.0	3213	9204.3	3249	9529.2	3284	9857.6
0.90	7189.3	2617	7451.0	2646	7715.6	2674	7983.0	2701	8253.1	2727	8525.8
0.95	6275.1	2195	6494.6	2217	6716.3	2238	6940.1	2258	7165.9	2278	7393.7
1.00	5489.5	1851	5674.6	1868	5861.4	1883	6049.7	1900	6239.7	1914	6431.1
1.10	4233.7	1337	4367.4	1348	4502.2	1357	4637.9	1367	4774.6	1375	4912.1
1.20	3301.6	986	3400.2	992	3499.4	998	3599.2	1004	3699.6	1010	3800.6
1.30	2603.8	740	2677.8	745	2752.3	748	2827.1	752	2902.3	756	2977.9
1.40	2076.0	565	2132.5	568	2189.3	571	2246.4	574	2303.8	575	2361.3
1.50	1672.4	439	1716.3	440	1760.3	442	1804.5	444	1848.9	446	1893.5
1.60	1360.5	345	1395.0	346	1429.6	348	1464.4	348	1499.2	350	1534.2
1.70	1116.9	275	1144.4	275	1171.9	277	1199.6	277	1227.3	278	1255.1
1.80	924.69	2215	946.84	2220	969.04	2226	991.30	2233	1013.6	224	1036.0
1.90	771.63	1803	789.66	1807	807.73	1812	825.85	1816	844.01	1821	862.22
2.00	648.64	1482	663.46	1485	678.31	1489	693.20	1491	708.11	1495	723.06
2.10	548.98	1228	561.26	1232	573.58	1234	585.92	1236	598.28	1238	610.66
2.20	467.58	1027	477.85	1029	488.14	1031	498.45	1033	508.78	1035	519.13
2.30	400.62	865	409.27	866	417.93	868	426.61	869	435.30	871	444.01
2.40	345.14	734	352.48	734	359.82	736	367.18	737	374.55	737	381.92
2.50	298.89	625	305.14	627	311.41	628	317.69	628	323.97	629	330.26
2.60	260.08	537	265.45	538	270.83	539	276.22	539	281.61	540	287.01
2.70	227.34	464	231.98	464	236.62	465	241.27	465	245.92	466	250.58
2.80	199.57	403	203.60	402	207.62	403	211.65	404	215.69	404	219.73
2.90	175.90	350	179.40	351	182.91	352	186.43	351	189.94	352	193.46
3.00	155.62	307	158.69	307	161.76	307	164.83	308	167.91	308	170.99

TABLE III—*Continued*

λ (microns)	7500°K		7600°K		7700°K		7800°K		7900°K		8000°K
	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ
0.26	20136.	2045	22181.	2190	24371.	2343	26714.	2500	29214.	2663	31877.
0.27	21890.	2136	24026.	2282	26308.	2431	28739.	2587	31326.	2746	34072.
0.28	23500.	2208	25708.	2351	28059.	2498	30557.	2648	33205.	2803	36008.
0.29	24952.	2261	27213.	2400	29613.	2542	32155.	2688	34843.	2838	37681.
0.30	26239.	2296	28535.	2430	30965.	2567	33532.	2708	36240.	2850	39090.
0.31	27358.	2314	29672.	2444	32116.	2574	34690.	2708	37398.	2845	40243.
0.32	28309.	2318	30627.	2442	33069.	2566	35635.	2694	38329.	2822	41151.
0.33	29097.	2309	31406.	2426	33832.	2545	36377.	2665	39042.	2787	41829.
0.34	29729.	2289	32018.	2399	34417.	2512	36929.	2625	39554.	2740	42294.
0.35	30213.	2258	32471.	2364	34835.	2468	37303.	2576	39879.	2683	42562.
0.36	30559.	2220	32779.	2319	35098.	2418	37516.	2518	40034.	2619	42653.
0.37	30777.	2176	32953.	2267	35220.	2361	37581.	2455	40036.	2548	42584.
0.38	30879.	2125	33004.	2212	35216.	2298	37514.	2387	39901.	2473	42374.
0.39	30874.	2071	32945.	2152	35097.	2233	37330.	2314	39644.	2396	42040.
0.40	30775.	2013	32788.	2089	34877.	2164	37041.	2240	39281.	2315	41596.
0.41	30592.	1953	32545.	2023	34568.	2093	36661.	2164	38825.	2235	41060.
0.42	30333.	1891	32224.	1957	34181.	2022	36203.	2088	38291.	2152	40443.
0.43	30009.	1829	31838.	1889	33727.	1950	35677.	2011	37688.	2072	39760.
0.44	29629.	1765	31394.	1822	33216.	1879	35095.	1935	37030.	1990	39020.
0.45	29200.	1702	30902.	1755	32657.	1807	34464.	1860	36324.	1912	38236.
0.46	28729.	1640	30369.	1689	32058.	1737	33795.	1786	35581.	1834	37415.
0.47	28224.	1578	29802.	1624	31426.	1669	33095.	1714	34809.	1758	36567.
0.48	27690.	1518	29208.	1560	30768.	1602	32370.	1643	34013.	1685	35698.
0.49	27134.	1458	28592.	1498	30090.	1537	31627.	1575	33202.	1613	34815.
0.50	26559.	1401	27960.	1437	29397.	1474	30871.	1509	32380.	1544	33924.
0.51	25971.	1345	27316.	1379	28695.	1411	30106.	1445	31551.	1478	33029.
0.52	25373.	1291	26664.	1322	27986.	1352	29338.	1384	30722.	1413	32135.
0.53	24770.	1237	26007.	1267	27274.	1296	28570.	1324	29894.	1351	31245.
0.54	24163.	1186	25349.	1214	26563.	1241	27804.	1267	29071.	1292	30363.
0.55	23555.	1138	24693.	1163	25856.	1188	27044.	1212	28256.	1236	29492.
0.56	22949.	1091	24040.	1114	25154.	1137	26291.	1160	27451.	1181	28632.
0.57	22348.	1045	23393.	1067	24460.	1088	25548.	1109	26657.	1130	27787.
0.58	21751.	1002	22753.	1022	23775.	1042	24817.	1061	25878.	1080	26958.
0.59	21162.	960	22122.	979	23101.	997	24098.	1015	25113.	1033	26146.
0.60	20581.	920	21501.	938	22439.	954	23393.	971	24364.	988	25352.
0.61	20010.	882	20892.	897	21789.	914	22703.	929	23632.	944	24576.
0.62	19449.	845	20294.	859	21153.	875	22028.	889	22917.	903	23820.
0.63	18899.	809	19708.	824	20532.	837	21369.	851	22220.	864	23084.
0.64	18360.	776	19136.	789	19925.	801	20726.	815	21541.	826	22367.

TABLE III—*Concluded*

λ (microns)	7500°K		7600°K		7700°K		7800°K		7900°K		8000°K
	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ	Δ	J_λ
0.65	17833.	744	18577.	756	19333.	767	20100.	780	20880.	791	21671.
0.66	17318.	713	18031.	725	18756.	735	19491.	747	20238.	757	20995.
0.67	16816.	684	17500.	694	18194.	705	18899.	714	19613.	725	20338.
0.68	16327.	655	16982.	666	17648.	675	18323.	685	19008.	693	19701.
0.69	15850.	629	16479.	638	17117.	647	17764.	656	18420.	664	19084.
0.70	15386.	603	15989.	612	16601.	620	17221.	629	17850.	636	18486.
0.71	14935.	579	15514.	586	16100.	595	16695.	602	17297.	610	17907.
0.72	14496.	556	15052.	563	15615.	570	16185.	577	16762.	585	17347.
0.73	14070.	533	14603.	541	15144.	547	15691.	553	16244.	561	16805.
0.74	13657.	511	14168.	519	14687.	525	15212.	531	15743.	537	16280.
0.75	13255.	492	13747.	497	14244.	504	14748.	510	15258.	515	15773.
0.80	11423.	402	11825.	407	12232.	412	12644.	416	13060.	420	13480.
0.85	9857.6	3319	10189.	336	10525.	338	10863.	342	11205.	344	11549.
0.90	8525.8	2753	8801.1	2777	9078.8	2802	9359.0	2825	9641.5	2848	9926.3
0.95	7393.7	2298	7623.5	2316	7855.1	2334	8088.5	2352	8323.7	2370	8560.7
1.00	6431.1	1930	6624.1	1943	6818.4	1958	7014.2	1970	7211.2	1984	7409.6
1.10	4912.1	1385	5050.6	1393	5189.9	1401	5330.0	1409	5470.9	1417	5612.6
1.20	3800.6	1016	3902.2	1020	4004.2	1026	4106.8	1030	4209.8	1035	4313.3
1.30	2977.9	759	3053.8	763	3130.1	766	3206.7	769	3283.6	771	3360.7
1.40	2361.3	578	2419.1	581	2477.2	582	2535.4	585	2593.9	586	2652.5
1.50	1893.5	447	1938.2	449	1983.1	450	2028.1	451	2073.2	453	2118.5
1.60	1534.2	351	1569.3	352	1604.5	353	1639.8	354	1675.2	355	1710.7
1.70	1255.1	279	1283.0	280	1311.0	281	1339.1	281	1367.2	281	1395.3
1.80	1036.0	225	1058.5	224	1080.9	226	1103.5	226	1126.1	226	1148.7
1.90	862.22	1824	880.46	1828	898.74	1832	917.06	1836	935.42	1839	953.81
2.00	723.06	1498	738.04	1501	753.05	1503	768.08	1507	783.15	1508	798.23
2.10	610.66	1241	623.07	1243	635.50	1245	647.95	1247	660.42	1249	672.91
2.20	519.13	1036	529.49	1038	539.87	1040	550.27	1041	560.68	1042	571.10
2.30	444.01	872	452.73	873	461.46	875	470.21	876	478.97	876	487.73
2.40	381.92	739	389.31	740	396.71	741	404.12	742	411.54	742	418.96
2.50	330.26	630	336.56	631	342.87	632	349.19	632	355.51	633	361.84
2.60	287.01	540	292.41	542	297.83	541	303.24	543	308.67	543	314.10
2.70	250.58	466	255.24	467	259.91	467	264.58	468	269.26	468	273.94
2.80	219.73	404	223.77	405	227.82	405	231.87	405	235.92	406	239.98
2.90	193.46	352	196.98	353	200.51	353	204.04	353	207.57	354	211.11
3.00	170.99	309	174.08	308	177.16	309	180.25	309	183.34	309	186.43

ON CERTAIN REDUCTIONS IN THE FOUR COLOR PROBLEM

By C. E. Winn

It was proved by Reynolds[1] that an irreducible map must contain at least 28 regions, while on the other hand he constructed a map of 30 regions free from any known reducibility.[2] Actually the latter comprises 6 heptagons each in contact wih 4 consecutive pentagons. Our first result is to reduce

(A) *A heptagon flanked by 4 consecutive pentagons.*[2a]

Consequently Reynold's map is no longer irreducible.

We next note the reducibility of Franklin,[3] namely

(B) *A polygon bounded by an even number of pentagons together with* (a) *two other adjacent polygons or* (b) *one other polygon.*

A similar reduction is found here of

(C) *A polygon bounded by two even sequences of pentagons, a hexagon and another polygon.*[4]

The case B was extended by Errera[5] to such a ring enclosing more than one polygon, provided no isthmus occurs in the reduced figure. A method is here given of reducing *without restriction*[6]

(D) *Any configuration bounded by the rings $B(a)$ and $B(b)$.*

Three other reductions follow, the first being due to Mr. I. Ratib:

(E) *A hexagon surrounded by the ring n55665.*

(E_1) *A hexagon surrounded by the ring mn5565.*

(F) A *hexagon in contact with two triads of pentagons.*

[1] "On the problem of coloring maps in four colors," Annals of Mathematics, Vol. 28 (1926) p. 1. Cf. also Franklin's abstract 290, Bull. A. M. S., Vol. 42 (1936), p. 491.

[2] "Some new methods of approaching the four color problem," Proceedings West Virginia Ac. of Science, I (1927) p. 91.

[2a] This result has been obtained independently by Mr. H. Chojnacki.

[3] "The four color problem," American Journal of Mathematics, Vol. 44 (1922), p. 225.

[4] A similar ring about any configuration can be reduced by the construction of Errera—loc. cit. (5)—in absence of an isthmus in the reduced figure.

[5] "Une Contribution au Problème des Quatre Couleurs," Bulletin de la Société Mathématique de France, Vol. 53 (1925), p. 42.

[6] The present result was obtained independently by Mr. Ratib and communicated by us at the Oslo Conference, 1936.

The latter case is only new when the contacts are separate, as otherwise it falls under

(G) *A pentagon or hexagon flanked by* 3 *consecutive pentagons.*[7]

It has recently[8] been shewn that

(H) *In an irreducible map a minor polygon (of* 5 *or* 6 *sides) must touch a major polygon (of more than* 6 *sides).* We prove now that

(J) *In an irreducible map a major polygon must touch either another major polygon or else* 3 *hexagons—or* 4 *hexagons in the case of a heptagon.*

Certain of the foregoing results simplify, and perhaps may amplify, the analysis of Reynolds above mentioned, as well as that of the necessary contacts of pentagons in an irreducible map.[9]

In conclusion we investigate an allied problem, shewing that:[10]

(K) *An irreducible map contains at least* 6 *major polygons.*

In order to allow for the possibility of an isthmus in our reduced figures, we require the reducibility of a "Birkhoff ring," i.e. *consisting of* 5 *polygons not enclosing a pentagon.*[11]

We have also occasion to quote the following reductions:

(L) *A polygon bounded by hexagons and pairs of pentagons, except an odd polygon bounded by hexagons only.*[12]

(M) *An even polygon enclosed by a ring L in which one hexagon is replaced by a pentagon,*[13] e.g. 55665666.

(N) *A pentagon flanked by* 4 *minor polygons of which the extremes are pentagons.*

The last case is obtained by combining *H* with that of the ring *n*5665 about a pentagon.[14]

We now proceed with our reductions, employing in more complicated cases a tabular scheme explained elsewhere.[8]

[7] G. D. Birkoff, "The Reducibility of Maps," American Journal of Mathematics, Vol. 35 (1913), p. 116, and P. Franklin, loc. cit. (3).

[8] C. E. Winn, "A case of coloration in the four color problem," American Journal of Mathematics.

[9] Loc. cit. (1), p. 477.

[10] This is an extension of the result of Errera, loc. cit. (5), that an irreducible map contains at least one major polygon.

[11] Loc. cit. (7).

[12] Franklin, loc. cit. (3).

[13] Loc. cit. (8), the generalised form of reduction H.

[14] Loc. cit. (8), reduction E.

A *mnp*5555 See fig. 1.

(1) $c = 4$		$u = 1$	
(2) $c = 2$ e or $g = 2$		$u = 1$	
(3) $e = 3, g = 4$ ($c = 2$ here as after)			
32 e to c	$d = 4$		
{13 h to f	$g = 2$		
(32 e to c	$d = 1$	$u = 1$	$e = 2, g = 2$ or 3
		$u = 1$)	$h = 3, b = 3$ or 1
		$u = 4$}	$e = 2$
		$u = 1$	
(4) $e = g = 4$			
42 g to a or c	$h = 3, b = 3$ or 1	$u = 4$	$g = 2$
		$u = 1$	
(5) $e = g = 3$			
32 e to g	$f = 4$		
{24 a to c or f	$b = 1$ or $gh = 13$	$u = 3$ or 1	$a = 4$
14 f to d	$e = 2$		
(12 h to c	$ab = 34$	$u = 1$	$h = 2$
24 h to c or f	b or $g = 1$	$u = 3$ or 1	$ha = 42$
		$u = 4$)	
14 f to h	$g = 2$		
(12 h to c	$ab = 34$	$u = 1$	$gh = 12$
41 a to g	$h = 3$	$u = 1$	
41 a to d	$bc = 23, fg = 14$	$u = 1$	$a = 1$
21 c to a	$b = 4$	$u = 3$	$gha = 212$
		$u = 1$)	$f = 1$
		$u = 1$}	$e = 2, a = 2$ or 3
		$u = 1$	

C $n\dot{5}\dot{5}6\dot{5}\dot{5}$. See fig. 2.[15]

If 2 does not occur at p, q, r or u, t, we put $x = 1$, $c = e = g = 2$, $k = i = 2$, and fill in the intervening polygons. Otherwise suppose $r = 2$, say. We take $c = e = 2$, filling in b, d as before. Then we color the rest of the ring from l, *so as to pass through the hexagon.*

[15] The recurring points indicate the possible repetition or absence of the sequence over which they occur.

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In proving D, we use the reduced figures of Errera.[16] We shall first suppose the absence of a smaller ring of type (a) or (b), i.e. with fewer polygons than the given ring Δ. Next, we observe that in case (b), by

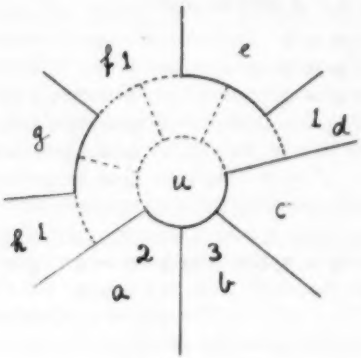

fig 1

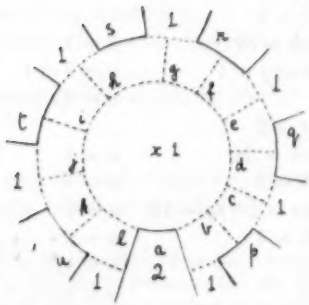

fig 2

reducing the side of Σ next to P_1 or $P_n (n \geqq 4)$, we ensure that a dotted line crosses Δ. Consequently, in view of our assumption,

1°. *P_1 and P_n do not coalesce in the reduction of* (b).

2°. *In both reductions, if P_i coalesces with $P_j (j > i + 1)$, it also coalesces with $P_{i+1}, P_{i+2}, \cdots, P_{j-1}$.*

[16] Loc. cit. (5), figs. 2 to 4 bis.

Hence no isthmus can occur in either figure at a dotted line, nor, in case (*b*), at an unreduced side between Σ and P_1 or P_n.

Further, should an isthmus appear at an unreduced side between Σ and Π, in case (*b*), it follows from 2° that Π forms a smaller ring than Δ with Σ and, as an examination of the figure shews, an *even* sequence of P's. Such a ring is, however, excluded by hypothesis.

The remaining possibility of an isthmus is at a side not abutting the *P*'s and lying between two polygons Ω, Ω' which have two *separate* contacts with the *P*'s. Again we find an even number of pentagons connecting Ω, Ω', as long as the conclusion of 2° is satisfied. This yields a similar contradicton unless the ring so formed is $\Omega P_1 P_2 \cdots P_n \Omega'$. But then the absence of a smaller ring of type (*b*) implies a polygon Λ touching all the *P*'s opposite Ω, Ω', and therefore a Birkhoff ring $\Omega P_1 \Lambda P_n \Omega'$.

Thus the given map, if irreducible, must contain a smaller ring Δ' of the same type as Δ. Likewise Δ', if not reducible, gives rise to a smaller ring Δ'' and so on, until we arrive at a Birkhoff ring.

*E n*55665 about a hexagon. See fig. 3.

(1) Unless $d = 2, f = 4, h = 3$ or 4 $\quad u = 1$

(2) $d = 2, f = 4, h = 3$

32 h to a	$i = 4$		
{24 a to d	$bc = 13$		
(23 a to c	$b = 4$	$u = 3$	
23 a to h	$i = 1, cd = 23$	$u = 1$	$a = 3$
		$u = 1$)	
24 a to f	$gh = 31$	$u = 1$	$ai = 42$
		$u = 3$}	$h = 2, d = 2$ or 3
		$u = 1$	

(3) $d = 2, f = 4, h = 4$

42 h to a	$i = 3$		
{32 d to b	$c = 4$	$u = 4$	$d = 3$
24 a to h	$i = 1$	$u = 1$	
24 a to f	$ghi = 321$		
(34 b to d	$c = 2$	$u = 3$	
34 b to f	$d = 4$	$u = 1$	$b = 4$
		$u = 1$)	$b = 4$
		$u = 1$}	$h = 2, d, f = 2$ or 4
		$u = 1$	

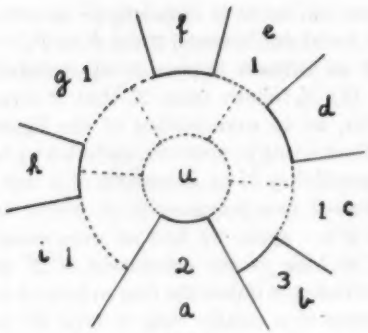

fig 3

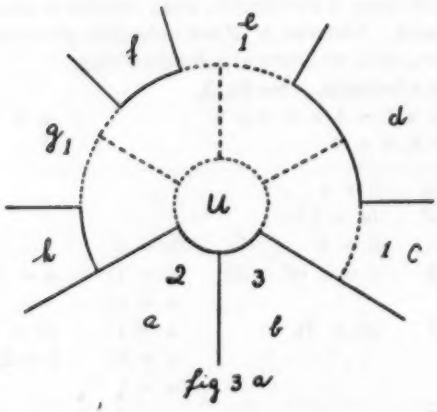

fig 3 a

E_1 *mn*5565 about a hexagon. See fig. 3a.
(1) Unless $d = 2$ or 4, $f = 2$, $h = 3$. $u = 1$
(2) $d = f = 2$, $h = 3$.

34 h to b	$a = 1$		
(32 h to b	$a = 4$	$u = 1$	
23 f to d	$e = 4$	$u = 2$	
32 h to d	$abc = 424$	$u = 1$	$h = 2, f = 2$ or 3
		$u = 2$)	$h = 4$
		$u = 1$	

(3) $d = 4, f = 2, h = 3$

34 h to d	$gfe = 212$	$u = 4$ $u = 1$	$h = 4, b = 3$ or 4

F $m5(5)5n5(5)5$ about a hexagon. See fig. 4[17]

(1) $b \neq d$ or $f \neq h$ $\quad u = 2, 3$ or 4
(2) $b = d = h = f = 2$ $\quad u = 2$
(3) $b = d = 2, f = h = 3, c = 4$ (or $g = 4$)

23 b to h	$a = 4$	$u = 3$ (1) or (2)	$b = 2$

(4) $b = d = 2, f = h = 3, c = 3, g = 2$

24 b to d	$c = 1$		
(23 b to d	$c = 4$	(3)	$b = 3$[18]
34 b to f	$cde = 212$	$u = 4$	$b = 4, h = 3$ or 4
		$u = 3$)	$b = 4, g = 2$ or 4
		$u = 3$	

In order to establish the first part of J, it will suffice, in view of B and L, to reduce a polygon bounded by a minor ring containing two hexagons and one or two *odd* sequences of pentagons.

$\dot{6}\dot{5}\dot{5}\dot{6}\dot{5}\dot{5}\dot{5}$ See figs. 5 and 6.

We adopt the reduction of fig. 5 or 6, according as there is one or more pentagons in the odd sequence. It is clear that, however p and q are colored, we can put 1 for x and either 324 or 423 for abc in fig. 5 and for hia in fig. 6. The rings can then be filled in as in C.

$\dot{6}\dot{5}\dot{5}\dot{5}\dot{6}\dot{5}\dot{5}\dot{5}$. See fig. 7.

Whether $p = 1$ or 3, we here put $x = 1, b = j = 3$, and complete the ring as in C.

The second part of J depends on the reductions of the rings 5555666, 5556566 and 5565656 about a heptagon that contain 3 hexagons. The first being covered by L, we now treat the other two cases.

[17] The pentagons in brackets are "caps" of the ring having contacts with the two regions between which they occur. I have since reduced the figure without the caps.

[18] The inversion 232 of fgh is symmetrically equivalent.

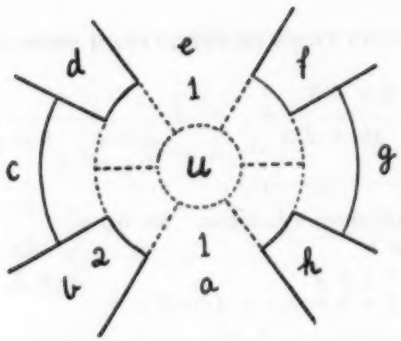

fig 4

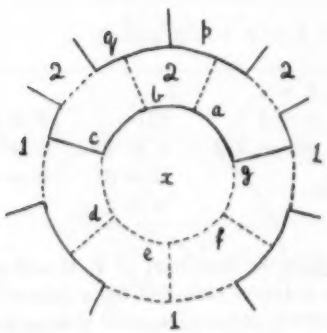

fig 5

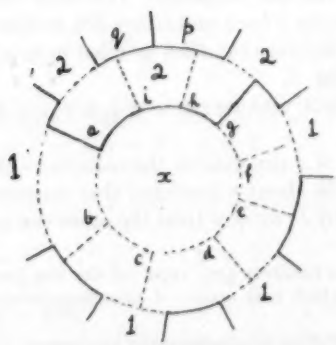

fig 6

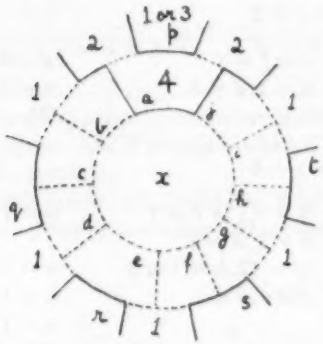

fig 7

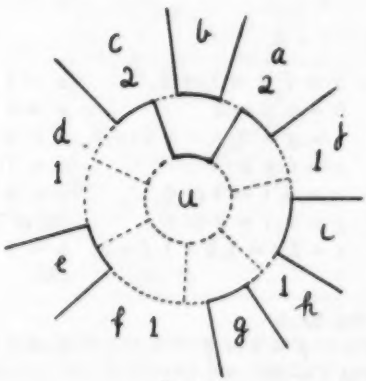

fig 8

5556656. See fig. 8.

(1) $b = 1$, unless $e = g = i = 2$ $u = 1$

(2) $b = 1, e = g = i = 2$

23 e to c or a	$d = 4, b = 1$ or 4	$u = 2$ $e = 3$ (1)

(3) $b = 3$, unless $egi = 422, 244$ or 444 $u = 1$ or 2

(4) $b = 3, e = 4, g = i = 2$

42 e to c or g	d or $f = 3$	$u = 2$
42 e to a	$b = 1, d = 3$	$u = 1$ $e = 2, i = 2$ or 4 (3)

(5) $b = 3, e = 2, g = i = 4$

24 c to a or i	$b = 1, j = 1$ or 3	$u = 1$
24 e to a	$b = 1, d = 3$	
(34 d to g or i	$ef = 12, h = 1$ or 2	$u = 1$ or 2 $d = 4$
41 d to b or f	c or $e = 3$	$u = 1$ $d = 1$ $u = 1$)
24 e to i	$e = 4, i = 2, g = 4$ or 2	(3) or (4) $c = e = 4$, $a, g = 4$ or 2 $u = 4$

(6) $b = 3, e = g = i = 4$

24 a to c or e	$b = 1, d = 1$ or 3	$u = 1$
24 c to i	$b = 1, j = 3$	$u = 1$
42 e to g	$e = g = 2$; $c, i = 2$ or 4	$u = 2$
42 e to i	$c = i = 2$	$u = 1$
31 b to h or j	$a = 4, i = 4$ or 2	$u = 4$
13 f to h or j	$g = 2, i = 2$ or 4	(3) or (4)
42 e to c	$e = 2, c = 4, b = 1, f = 3$	$u = 1$ $e = 2$ (5)

5565656. See fig. 9.

(1) $b = 1$, unless $e = g = i = 2$ $u = 1$

(2) $b = 1, e = g = i =, 2$

23 e to c	$d = 4$	$u = 2$, $e = 3$; $a, i, g = 2$ or 3 $u = 1$

(3) $b = 3$, unless $egi = 232$, 434 or 444 $u = 1$ or 2

(4) $b = 3, egi = 232$

32 g to e (or i)	$f = 4$	$u = 2$ $g = 2, c = 2$ or 3 $u = 2$ or 1

(5) $b = 3, egi = 434$

43 e to g or i	$f = 2, h = 1$ or 2	$u = 2$ $c = 3, b = 3$ or 4 $u = 1$

(6) $b = 3, e = g = i = 4$

31 b to d or f	$c = 4, e = 4$ or 2	$u = 4$ $b = 1$ $u = 1$

In proving K, we recall (D and J) that an irreducible minor ring must always contain 2 hexagons—and 3 hexagons when it surrounds a single polygon. We need the following

Lemma An irreducible minor ring containing 2 (*or* 3) *hexagons only, includes also* 2 (*or one*) *pairs of pentagons whose common side abuts two major polygons.*

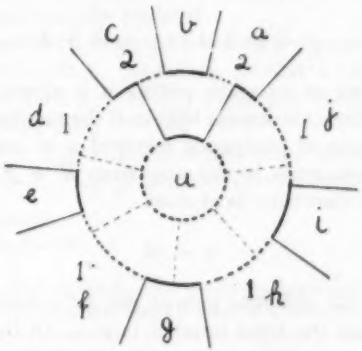

fig 9

We shall, in fact, establish the existence of one or two sequences of 3 pentagons in the given ring R whence the required pairs are derivable on account of N and G.

First, when R bounds a single polygon of n sides, we may take $n > 7$ (H and J). If $n > 9$, we get 3 consecutive pentagons, as desired. Such is also the case when $n = 8, 9$, except for the rings 55655656 and 556556556, which, however, are reducible by M and L.

Secondly, let R enclose a configuration S of more than one polygon. If n, the number of regions in R, exceeds 9, we evidently have one or two sequences of 3 pentagons, according as there are 3 or 2 hexagons in R. If $n \leq 9$, it follows from H that S contains one major polygon of 7 or 8 sides. By J a heptagon of S touches 4 hexagons, of which one belongs to S. Moreover, the latter is reducible by G, unless it touches another

hexagon; and this for the same reason must belong to S. Consequently, $n > 9$! A similar contradiction is obtained at once for an octagon of S, to complete the proof.

If possible, let an irreducible map contain 5 major polygons having a total of p vertices (repeated, if common).

First, suppose that no 4 major polygons form a chain, i.e. part of a ring. We then group contiguous major polygons—together with a pentagon, should it touch 3 of them along adjacent sides.[19] A configuration S so formed is simply-connected and bounded by a ring R, in absence of a Birkhoff ring. Moreover, if S consists of k major polygons with a total of q vertices (repeated, if common), R contains at most

$$q - 4k + 4 \text{ or } q - 4k + 3$$

polygons, according as an extra pentagon is absent or present in S.

Now it follows from the lemma that in R the number of hexagons plus the number of pairs of pentagons required is at least 4. Hence the contribution of pentagons to the map from $R + S$ (allowing for the repetition of pairs elsewhere) is at most

$$q - 4k$$

Summing over the different groups, we get a maximum of $p - 20$ pentagons, whereas the total number is $p - 18$ by Euler's relation. Consequently, two remaining pentagons are reducible under H.

Next, if 4 major polygons uwx form a chain, separate from the fifth one y, we include in the configuration S:

1° A pentagon adjacent to uw and/or vwx

2° A pentagon adjacent to awx (or auv), when a touches uvw (or vwx)

3° in absence of 1° a pentagon b touching uwx and a pentagon bounding ubx.

4° in absence of 1°, 3° a hexagon adjacent to $uvwx$.

In order to prove that S is now surrounded by a ring, we observe that a 5-ring $uwvcd$, $vwxef$ or $uwxfg$ encloses a pentagon of 1° or 3°. Further a ring $uwxhi$ encloses either two pentagons or a hexagon on the side remote from y, as in 1° and 2°, 3° or 4°. For, in view of H, the polygon j adjacent to hi on this side must meet uv, wx, ux or uwx—

[19] A second pentagon cannot have triple contact, or one of the major polygons would possess less than 6 vertices.

otherwise a Birkhoff ring would occur.[20] Finally, any other contact preventing a ring round S, e.g. $uklw$, involves a ring of 4 or less. So we can safely apply our previous argument to the ring surrounding S, obtaining a similar contradiction.

Lastly, let $uwxy$ be connected, uwx forming a chain. Should $uwxy$ complete a ring, they enclose a pentagon and are surrounded by a second ring, to which the earlier proof applies. (In fact, the second ring has $p - 20$ contacts only.)

If $uwxy$ is a chain, it has $p - 16$ contacts, 3 occurring at each of u, w, y; and it follows from N that u, w, y each touch a hexagon. This leaves $p - 19$ contacts at S, as against $p - 18$ pentagons in the map, so that a pentagon is reducible under H.

If $uwxy$ form neither a ring nor a chain, S has $p - 17$ contacts or less, 3 occurring at u and x. As before, we find a hexagon adjacent to u and x, leaving at most $p - 19$ other contacts at S. Thus the given map is in all cases reducible.

The reduction of a map with less than 5 major polygons is virtually a special case of the above, and presents no further difficulty.

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[20] It can be shewn that an irreducible minor configuration (simply connected) having 6 contacts at its boundary must consist of a hexagon or a pair or triad of pentagons. In the present case, if j were a pentagon of a triad, its contacts with hi would make it reducible by H. It can also be shewn that an irreducible minor configuration with 7 contacts at its boundary is composed of at most 4 polygons.

NOTE ON THE FOUR COLOR PROBLEM[1]

By Philip Franklin

1. **Introduction.** In 1920, using the known reducible configurations for the four color problem due to A. B. Kempe[2] and G. D. Birkhoff,[3] together with a few additional ones, I proved[4] that every map on a sphere containing 25 or fewer regions can be colored in four colors. Five years later C. N. Reynolds[5] succeeded in replacing the number 25 by 27, and gave an example which showed that the number could not be raised further without using additional reductions. He was able to simplify his argument by employing certain reductions due to A. Errera.[8]

Recently, I obtained a few reducible configurations, which enabled me to raise the number to 31. However, before these were published, C. E. Winn[6] independently obtained a number of new reductions which include most of mine as special cases. He[7] also established the general result that any map containing at most one polygon of more than 6 sides can be colored. This goes far beyond Errera's[8] theorem that any map containing no polygons of more than 6 sides is reducible.

In this note, using the reductions of Winn to shorten my argument, I shall prove that every map on a sphere containing not more than 31 regions can be colored. At the end of the note are given a few reductions not included as special cases of those found by Winn, and a generalization of a result due to Heawood.[9]

2. **Notation.** We follow to some extent the notation of Reynolds,[5] using capital letters to denote certain types of configurations, and the

[1] Presented to the American Mathematical Society.
[2] Kempe, A. B., Am. Journal of Math. 2 (1879) p. 193.
[3] Birkhoff, G. D., Am. Journal of Math. 35 (1913) p. 115.
[4] Am. Journal of Math. 44 (1922) p. 225.
[5] Reynolds, C. N., Annals of Math. 28 (1926) p. 1.
[6] Winn, C. E., Journal of Math. and Physics 16 (1937), p. 159.
[7] Winn, C. E., Amer. Jour. of Math. (1937).
[8] Errera, A., Bull. de la Soc. Math. de France 53 (1925), p. 42.
[9] Heawood, P. J., Proc. London Math. Soc. 40 (1935), p. 189.

corresponding small letters to denote the number of configurations in an irreducible map.

A shall denote a region of our map.

A_n shall denote a region of n sides.

P_1 shall denote a pentagon in contact with no other pentagon.

P_2 shall denote a pentagon in contact with one other pentagon.

P_3 shall denote a pentagon meeting two other pentagons at a point.

P_4 shall denote a pentagon meeting two other pentagons along non-consecutive edges.

P_5 shall denote a pentagon in contact with three other pentagons.

V shall denote a vertex of our map.

E shall denote an edge of our map.

V_0 shall denote a vertex of our map which is not the vertex of any pentagon.

3. **Relations for irreducible maps.** We recall some known relations. We have[2, 3]

$$3v = 2e = \sum_5 n a_n, \qquad a = \sum_5 a_n \tag{1}$$

by counting ends of edges, and recalling that in an irreducible map all vertices are triple and all polygons have at least five sides. The Euler relation

$$v - e + a = 2 \tag{2}$$

may be combined with this to give

$$a = 2 + \frac{v}{2}, \tag{3}$$

and

$$a_5 = 12 + \sum_7 (n - 6)a_n . \tag{4}$$

We next[4, 5] count the contributions of vertices for the pentagons of the various types possible in an irreducible map, dividing the contribution equally when two or three pentagons meet at a point. This leads to

$$v = 5p_1 + 4p_2 + 3\,1/3p_3 + 3p_4 + 2\,1/3p_5 + v_0 . \tag{5}$$

We shall in the sequel principally use (5) combined with (3).

4. **Any irreducible map must contain 32 regions.** Since a is an integer, if we show that for an irreducible map v must exceed 58, it will

follow from (3) that a is at least 32. We start with Winn's result[6] that an irreducible map must contain at least six polygons of more than six sides.

a) Suppose the map contains exactly 6 heptagons, and no other polygons of more than 7 sides, so that by (4) there are 18 pentagons. If all these pentagons are of type P_1, P_2 or P_3, by (5)

$$v \geqq 18 \cdot 3\ 1/3 \geqq 60 > 58.$$

Since four consecutive pentagons flanking a heptagon is reducible,[6] we may not introduce this configuration. Thus, if a string of mP_4's is present, with $2P_3$'s at the ends, it will reduce the limit for v by

$$(m + 2)3\ 1/3 - [2 \cdot 4 + m \cdot 3] = \frac{m - 4}{3},$$

but will introduce m sets of 3 consecutive pentagons, each of which will have to be in contact with a separate heptagon.

Again, the only configuration involving P_5's which does not have four consecutive pentagons is a T shape, made up of 2 P_3's, one P_5 and a P_2. This will reduce the limit for v by

$$4 \cdot 3\ 1/3 - [2 \cdot 3\ 1/3 + 2\ 1/3 + 4] = 1/3,$$

but will introduce 2 sets of 3 consecutive pentagons, each of which will have to be in contact with a separate heptagon.

Thus, with only six separate heptagons available, the limit of 60 for v can not be reduced by more than unity, which leaves 59, still in excess of 58. This establishes our contention that the map has at least 32 regions in case (a).

b) Suppose the map contains more than 6 heptagons, but no polygons of more than 7 sides. We are again limited to the configurations mentioned in (a). With only 7 separate heptagons, the sum can be reduced by unity, either by 3 T's, or one sequence with $m = 7$, but not by more. Since there are now 19 pentagons, we have

$$v \geqq 19 \cdot 3\ 1/3 - 1 \geqq 62\ 1/3 > 58.$$

With 8 or more heptagons we must have at least 20 pentagons. The coefficients in (5) all exceed 3, with the exception of that of p_5. But the only admissible configuration involving a P_5, is the T for which the average coefficient is 13/4 which exceeds 3. Thus, we now have

$$v \geqq 20 \cdot 3 \geqq 60 > 58,$$

and the map again has at least 32 regions in case (b).

c) Suppose the map contains one octagon, 5 or more heptagons, and no other polygons of more than 7 sides. We then have at least 19 pentagons. Any configuration of pentagons made up of P_1, P_2, and P_4 only, leads to a contribution to the sum in (5) 2 greater than that with an average coefficient of 3, due to the two P_2 at the ends. Configurations of pentagons must have ends, as rings are reducible.

Configurations made up of P_2 only, give 1 greater than an average coefficient of 3, for each triad or separate figure.

Three P_5's at a point involve 3 sets of 4 consecutive pentagons, each of which will require a separate polygon of more than 7 sides, so that they are not admissible for this case.

A T consisting of two P_2's, one P_5 and a P_2, gives 1 greater than an average coefficient of 3, and uses up 4 pentagons.

An elongated T, T_m, with mP_4 inserted, involves a set of 4 consecutive pentagons. Thus only one can be present in this case, and we can not have a figure with such a T at each of its two ends. The T_m gives 1 more than an average coefficient of 3.

Finally a figure involving two P_5's meeting one P_2 at a point involves a set of 4 consecutive pentagons. It has two ends, neither of which can be closed by a T in this case, so that there are two P_2's in the figure. Such a configuration gives 1 more than an average coefficient of 3.

Recalling that 6 consecutive pentagons flanking an octagon form a reducible figure,[4] we see that the figure last discussed may be elongated at one end by the insertion of mP_4, but not at both ends. The figure will then contain $m + 1$ sets of 3 consecutive pentagons, in addition to the set of 5 flanking the octagon. Thus such a figure can not have m exceeding 4, in this case (c). For $m = 4$, the figure contains 9 pentagons.

Similar considerations may be applied to show that a T_m must contain at least $m - 1$ sets of 3 consecutive pentagons, in addition to the set of 5 flanking the octagon, or m sets of 3 in addition to the set of 4 flanking the octagon. Thus in this case (c), a T_m can not have m exceeding 6. For $m = 6$, the figure T_6 contains 10 pentagons.

The 19 or more pentagons must include either one figure giving 2 more than an average coefficient of 3, or at least two figures each giving 1 more than an average coefficient of 3. Thus in this case (c),

$$v \geqq 19 \cdot 3 + 2 \geqq 59 > 58,$$

and the map has at least 32 regions in case (c).

d) Suppose the map contains either one polygon of more than 8 sides, or two polygons of more than 7 sides, but not more than two such

polygons. Then, in the least favorable cases, $a_9 = 1$, $a_7 = 5$, or $a_8 = 2$, $a_7 = 4$; so that by (4) there are 20 pentagons. Thus in case (d) we always have at least 20 pentagons.

Configurations not involving P_5, lead to coefficients of 3 or more in the sum (5).

As in case (c), we can not have 3 P_3's meeting at a point. A P_5 meeting two P_3's at a point gives an average coefficient of 3 for these three pentagons. Two P_5's meeting a P_3 at a point gives a sum less by 1 than that given with an average coefficient of 3. But each such figure leads to a set of 4 consecutive pentagons, so that if in separate figures not more than two can be present in this case (d). If one of the ends of the connected set of pentagons involving one or more of the figures just described is a P_2, it gives a surplus of 1 above the average of 3. Thus we would have for the sum v,

$$v \geqq 20 \cdot 3 - 2 + 1 \geqq 59 > 58.$$

To avoid the surplus of 1, both ends would have to be pairs of P_3, with a single P_5, or additional sets must be present for P_2's at the ends. Such figures may be constructed, but any figure with a deficiency of 1, including at most one set of 4 or more consecutive pentagons, has a sequence of 8 consecutive pentagons, while those with a deficiency of 2, including at most one set of 4 or more consecutive pentagons, have a single sequence of 12 consecutive pentagons. Figures with a deficiency of 2, having exactly two sets of 4 or more consecutive pentagons, have either one single sequence of 10 consecutive pentagons, or two sequences of 7 consecutive pentagons.

Since in an irreducible map a sequence of 7 consecutive pentagons involves the presence of a nonagon, while a sequence of 8 consecutive pentagons involves the presence of an 11 sided figure in an irreducible map, in each of these cases the map contains a minimum of 22 pentagons, and

$$v \geqq 22 \cdot 3 - 2 \geqq 64 > 58,$$

so that we again have at least 32 regions.

Figures with higher deficiency involve more than one additional pentagon for each deficiency of 1, so that the conclusion that 32 is a minimum still holds for them in every case (d).

e) Suppose the map contains 3 or more polygons of more than 7 sides. Then, in the worst case, $a_8 = 3$, $a_7 = 3$, so that by (4) there are 21 pentagons, as a minimum number in case (e).

Since the only coefficient less than 3 in the sum (5) is that of p_5, we again consider figures involving P_5. A P_5 meeting two P_3's has an average coefficient of 3, for the three pentagons. Two P_5's meeting a P_3 at a point gives a deficiency of unity in the sum, but a sequence of 4 consecutive pentagons. Three P_5's meeting at a point gives a deficiency of 2, but involve 3 sequences of 4 consecutive pentagons.

If there are m sequences of 4, each involving a separate octagon, there are at least $12 + 2m$ pentagons, and a deficiency of at most m. If some of these sequences of 4 combine, a union of k figures each of which would separately give a sequence of 4, gives a sequence of at least $2k + 2$. That is, the ends of the intermediate sets of 4 are lost, but the central pairs of P_5's remain. But such a sequence of $2k + 2$ consecutive pentagons in an irreducible map involves a $2k + 5$ gon, which leads to $2k - 1$ pentagons in the sum (4), one less than if the sets were in contact with separate octagons.

If there are less than 5 sequences of 4, we have

$$v \geqq 21 \cdot 3 - 4 \geqq 59 > 58.$$

If there are m, $m \geqq 5$, separate sequences of 4, there are at least $12 + 2m$ pentagons, and at most a deficiency m, so that

$$v \geqq (12 + 2m)3 - m \geqq 36 + 5m \geqq 61 > 58.$$

If some of these sequences are combined, there may be a loss in the number of pentagons. For $m = 5$, we have used $12 + 2m = 22$ pentagons. If these were combined in 2 sequences, with $k = 2$, and one separate sequence of 4, the sequences of 6, 6, 4 would require polygons of 9, 9, 8 sides, but as there are now three other polygons of at least 7 sides, there are 23 pentagons, greater than the number used for the separate sequences. Similarly, the other possible combinations when $m = 5$, lead to more than 22 pentagons. In fact, we can lose at most 2 pentagons by combination, and will gain at least 2 pentagons from the necessity for a total of 6 polygons of at least 7 sides.

If there are 6 sequences, some of which are combined, we can lose at most 3 pentagons by combinations, and will gain at least 1 pentagon from the necessity for a total of 6 polygons of at least 7 sides. Thus, if $m = 6$, we will always have

$$v \geqq (12 + 2m)3 - m \geqq 60 > 58.$$

If there are m, $m > 6$, sequences, some of which are combined, we can lose at most $m/2$ pentagons by combinations, so that

$$v \geqq (12 + 2m - m/2)3 - m \geqq 36 + 7m/2 \geqq 60\ 1/2 > 58.$$

We again have at least 32 regions in case (e). As the five cases include all possible irreducible maps, we have established the theorem:

Every irreducible map must contain at least 32 *regions.*

As a consequence,

Every map on a sphere with at most 31 *regions is colorable in four colors.*

5. **Four New Reductions.** We here prove the reducibility of the following configurations.

A. *An odd sided region, surrounded by hexagons and one external pentagon,* 6(5)66 $\cdots$ 66.

B. *Two adjacent pentagons, each touching the same heptagon which together with five hexagons surrounds them.*

C. *An odd sided region surrounded by* 1*mn* 5(5)5 $\cdots$ 5(5)5.

D. *A polygon surrounded by* 5(5)76 $\cdots$ 6.

D_1. *In the case D, a pair of consecutive hexagons may be replaced by two pentagons.*

We follow the method of exposition used by Winn,[7] in proving these configurations reducible. Small letters refer to regions in the figure. The numbers 1, 2, 3, 4 refer to colors of regions. (1), (2), etc. refer to possible cases. Pairs of numbers as 32 b to d means assuming a 32 chain to join 3 and 2. We give on the right indications of the coloration of certain regions, stopping when the coloration may obviously be completed. The colorations on the left are assumptions defining the cases (1), (2), etc.

A. An odd sided region, surrounded by hexagons and one external pentagon. We illustrate for a heptagon Fig. 1.

(1) $\quad g = 2, f = 3 \qquad\qquad m = 1, l = 2$

By symmetry, we may treat all cases in which b, d, e, f, g, h, i are not all similarly colored.

(2) $\quad b = d = e = f = g = h = i = 2, \qquad a = 4$

32 b to d $\qquad\qquad c = 4, j = 3$

$\qquad\qquad\qquad\qquad\qquad\qquad m = 2, k = 1$

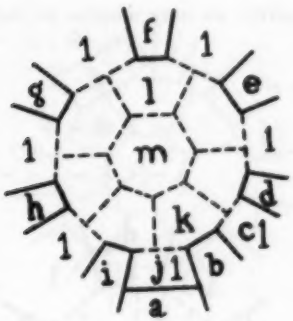

FIG. 1

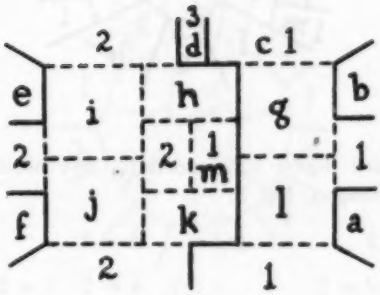

FIG. 2

B. Two adjacent pentagons, each touching the same heptagon and surrounded by this and the five hexagons. Fig. 2.

(1)	$b = 4$	$h = 4$
(2)	$b = 2, a = 3$	$h = 4, g = 3$
(3)	$b = 2, a = 4$	$h = 4, g = 3, l = 2$
(4)	$b = 3, a = 2$	$h = 4, g = 2$

There are four symmetric cases with b, a replaced by e, f.

(5)	$b = a = 2$ or 3 and $e = f = 1$ or 3	$j = 4, l = 4$

In view of the symmetry, we may assume as the only remaining possibility

(6) $b = 3, a = 4, e \neq 4.$

34 b to d $\quad c = 2, m = 3$

$i = 4, h = 1, g = 4, l = 2, k = 4$

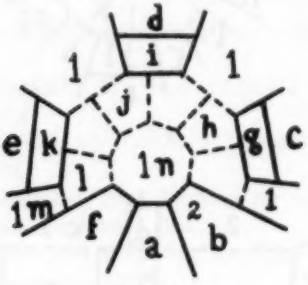

FIG. 3

C. An odd sided region surrounded by lmn 5(5)5 $\cdots$ 5(5)5. We illustrate for a nonagon, Fig. 3.

(1) $b = 2, c \neq 2$ $\quad g = 2$

(2) $b = c = 2, d \neq 2$ $\quad h = 2, \ i = 2$

(3) $b = c = d = 2, e \neq 2$ $\quad h = 2, j = 2, k = 2$

(4) $b = c = d = e = 2, f \neq 2$ $\quad h = 2, j = 2, l = 2$

(5) $b = c = d = e = f = 2, a = 4$

32 e to f $\quad m = 4, n = 3,$

$g = i = k = 3, l = 1$

D. A polygon surrounded by 5(5)76 $\cdots$ 6.
We illustrate the even case for an octagon, Fig. 4.

(1) $\quad a = 2 \quad c = 2$

(2) $\quad a \neq 2 \quad b = 2$

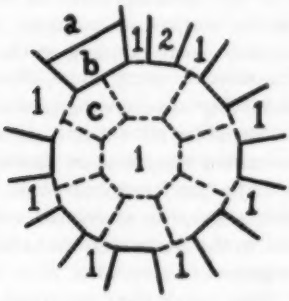

FIG. 4

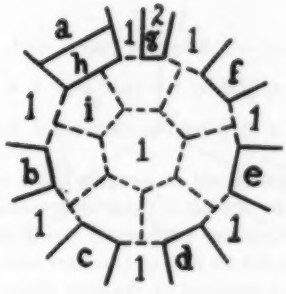

FIG. 5

We illustrate the odd case for a heptagon, Fig. 5.

(1)	$a = 2$	$i = 2$
(2)	$a = 3, b = c = d = e = f = g = 2$	$h = 4, i = 2$
(3)	$a = 3$ one of $b, c, d, e, f \neq 2, g = 2$	$h = 2$

D_1. One or more pairs of consecutive hexagons may be replaced by pentagons.

Proved as in D, merely omitting two lines and one region from the figures for each pair replaced.

6. **Heawood's Residuals.** The coloration of a given regular map on a sphere in four colors is equivalent to the solution of a certain set of

congruences,[10] modulo 3, by variables none of which are zero. The variables correspond to the vertices of our map, and each congruence is the sum of those variables corresponding to the vertices of a region of our map, set equal to zero.

In estimating the probability of such congruences having proper solutions, i.e. values of the variables all $\equiv 1$ or -1, modulo 3, Heawood[11] considered a set of generalized congruences, similar to those just mentioned except that the right hand members were not necessarily zero, but instead any numbers arbitrarily associated with the regions of the map, and only subjected to the restriction that the sum of the numbers for all the regions is congruent to zero, mod. 3.

In connection with these generalized congruences, Heawood[9] introduced three operations on a finite, ordered set of integers, each reduced mod. 3 to 1, -1 or 0.

1. Replacing a triplet $a, -1, b$ by the pair $a + 1, b + 1$.
2. Replacing a triplet $a, 1, b$ by the pair $a - 1, b - 1$.
3. Replacing a triplet $a, 0, b$ by the element $a + b$.

In particular, if the ordered set of integers has a sum zero, mod. 3, the same will be true of the set after these operations are performed any number of times. If they are performed a sufficient number of times, we shall eventually be led to one of the four sets 0, 0; 1, -1; -1, 1; or 0. The set 0, 0 is a residual of the first class; the other three sets are each residuals of the second class. By cyclically permuting the ordered set, we may change the residuals from 1, -1, to 0, but we can not change the class of residuals, which is also independent of the order in which the reducing operations are applied.

Suppose we draw a simple circuit on a map, separating the map into two portions, for one, and hence both, of which the numbers associated with the regions of the map, or right members of the generalized congruences add up to zero, mod. 3. Also, suppose we have proper solutions of the congruences corresponding to the regions on one side of the map. We may form a cyclically ordered set of integers for the regions abutting one side of the circuit, by adding up the numbers 1 or -1 corresponding to the vertices of such a region which are on the circuit.

[10] Heawood, P. J., Quart. Jour. of Pure and Applied Math. 29 (1897) p. 270
Veblen, O., Annals of Math. 14 (1913) p. 163.
[11] Heawood, P. J., Proc. London Math. Soc. 33 (1932) p. 253.

The sum of such a set will be zero, mod. 3, and we may determine the class of its residuals.

7. **Heawood's Results, and a generalization.** Heawood[9] proved that the class of residuals for a circuit of the kind just described is independent of the side of the circuit used to determine the abutting regions. In fact, if the right members of the generalized congruence are all zero for the regions of a ring bounded by two circuits of this kind, for any proper solution of the congruences the two circuits will have the same class of residuals. Also, a circuit of this type having on one side only regions for which the right members of the generalized congruence are zero, will necessarily have residuals of the first class, i.e. 0, 0.

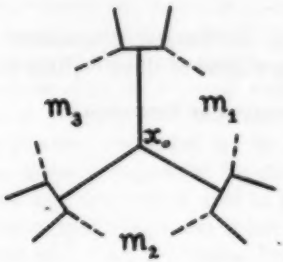

FIG. 6

Using these results, Heawood[9] deduced that:

I. If we associate the numbers 1 and -1 with two adjacent regions, and zero with all the remaining regions, the generalized congruences can not have any proper solutions.

II. If we consider the original congruences with right members zero, and find proper values solving all except possibly two which correspond to adjacent regions, these two will automatically be solved also.

We give here a slight extension of this, namely:

If we consider the original congruences with right members zero, and find proper values solving all except possibly three which correspond to the regions touching a vertex, then the variable for this vertex may be so chosen that all the congruences are satisfied.

We consider the circuit bounding the three regions. Since the vertex values solve the congruences for all the regions outside this circuit, the residuals obtained from them for the outside of the circuit will be 0, 0,

and the same will be true for the residuals obtained from them for the inside of the circuit. If the set of integers for the inside of the circuit are, Fig. 6, m_1, m_2, m_3, for the three regions, since

$$m_1 + m_2 + m_3 \equiv 0, \text{ mod. } 3$$

we must have either 0, 0, 0; 0, 1, −1; 1, 1, 1 or sets obtained from them by cyclic permutation or change of sign, neither of which changes the class of residuals. The only one of these three having residuals of the first class is 1, 1, 1. Thus, either m_1, m_2, m_3 are 1, 1, 1 and we set $x_0 = -1$, or m_1, m_2, m_3 are −1, −1, −1 and we set $x_0 = 1$. In either case the omitted congruences, which are essentially

$$m_i + x_0 \equiv 0, \qquad \text{mod. } 3,$$

will be satisfied.

The value of x_0 may be changed, if necessary, without affecting the other congruences, since none of these contain x_0.

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A THEOREM RELATING NON-VANISHING AND ANALYTIC FUNCTIONS

By Norman Levinson

The fact that an analytic function and all its derivatives cannot vanish at a point without vanishing identically has been generalized by Denjoy, Carleman, and others. Qualitatively stated, they have shown that if the derivatives of a function do not increase in magnitude too rapidly it has the above stated property of analytic functions. The functions they have introduced are known as quasi-analytic.

It is clear that an analytic or quasi-analytic function cannot vanish over an interval without vanishing identically. This more restrictive property of not vanishing over an interval without being identically zero does not, however, require the existence of derivatives. For example, an analytic function bounded on its circle of convergence cannot vanish over an interval unless the function is identically zero, but it need not possess any derivatives, and in general does not.

This property of non-vanishing over an interval has been shown to hold for functions $F(x)$ on $(-\infty, \infty)$ whose Fourier transforms $G(u)$ become sufficiently small either as $u \to \infty$ or as $u \to -\infty$, (but not necessarily both). In general, such an $F(x)$ will not possess derivatives, and therefore lies outside the scope of usual quasi-analytic theorems.

A typical result of the type just discussed is[1]

Let $G(u) \in L(-\infty, \infty)$ *and suppose that*

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(u) e^{-iux}\, du. \tag{1.0}$$

Let

$$G(u) = 0(e^{-\theta(u)}), \qquad u \to \infty, \tag{1.1}$$

where $\theta(u)$ *is a non-decreasing positive function such that*

$$\int_1^{\infty} \frac{\theta(u)}{u^2}\, du = \infty. \tag{1.2}$$

[1] N. Levinson, "On a Class of Non-vanishing Functions," Proc. London Math. Soc., Vol. 41, p. 393. Theorem II.

Then $F(x)$ cannot be equivalent to zero at any interval unless it is equivalent to zero over $(-\infty, \infty)$.

For certain applications it becomes necessary to consider a more general situation than that covered in the above result. Suppose $z = x + iy$ and $f(z)$ is analytic in $(0 \leqq x \leqq 1, 0 < y \leqq \gamma)$ and continuous in $(0 \leqq x \leqq 1, 0 \leqq y \leqq \gamma)$. Suppose $F(x)$ satisfies the conditions of the theorem quoted above and that $F(x) = f(x)$, $0 \leqq x \leqq \frac{1}{2}$. Does it follow that $F(x) = f(x)$ over $\frac{1}{2} \leqq x \leqq 1$? That it does in the special case $f(z) = 0$ is what the above result states. That it is true for any $f(z)$, it is our object to show.

$F(x)$ is an extension of the class of functions which have the real axis as a boundary of analyticity. This latter class would have $G(u) = 0$ as $u \to \infty$ instead of $G(u)$ being merely small as $u \to \infty$. How closely it is related to this class the following theorem shows:

Theorem I. Let $G(u) \in L(-\infty, \infty)$ *and let* (1.0), (1.1) *and* (1.2) *hold. Let $f(z)$ be analytic in $(a \leqq x \leqq c, 0 < y \leqq \gamma)$ and continuous in $(a \leqq x \leqq c, 0 \leqq y \leqq \gamma)$. Suppose $F(x) = f(x)$ for $a \leqq x \leqq b < c$. Then $F(x) = f(x)$ in $a \leqq x \leqq c$.*

Theorem I can be generalized in several directions. First (1.1) may be replaced by less restrictive conditions.[2] The condition

$$G(u) \in L(-\infty, \infty)$$

also can be considerably weakened. However, all these results can be obtained from Theorem I by devices frequently used in transform theory, and will therefore be omitted here.

2. We require several lemmas.

Lemma 1.[3] *If $M(u)$ is a positive, non-increasing function and if*

$$\int_1^\infty \frac{du}{u^2} \log \{e^{-u} + M(u)\} > -\infty$$

then

$$\int_1^\infty \frac{du}{u^2} \log M(u) > -\infty.$$

[2] Any of the conditions given in Levinson, loc. cit., are sufficient, and can be proved similarly.

[3] Levinson, loc. cit., Lemma 2. Here it is stated that $M(u)$ is also continuous, but it is trivial to show that if the theorem holds for continuous $M(u)$ it also holds for a discontinuous one.

Lemma 2. *If $f(z)$ is analytic in $(0 \leqq x \leqq 1, 0 < y \leqq \gamma < \frac{1}{2}\pi)$ and continuous for $0 \leqq y \leqq \gamma$ and if*

$$J_1(u) = \int_0^1 \left(\frac{e^x - 1}{e - e^x}\right)^{iu} (e^x - 1)^2(e - e^x)^2 f(x)\, dx \tag{2.0}$$

then

$$|J_1(u)| \leqq 10^5 e^{-\frac{1}{2}\gamma u} \operatorname{Max} |f(z)|, \qquad u > 0, \tag{2.1}$$

where in $\operatorname{Max} |f(z)|$, *$z$ ranges over $(0 \leqq x \leqq 1, 0 \leqq y \leqq \gamma)$.*

Proof: By the Cauchy integral theorem taking instead of the path $(y = 0, 0 < x < 1)$, the path made up of the arms of the iscosoles triangle $(y = \gamma x, 0 \leqq x \leqq \frac{1}{2})$ and $(y = \gamma(1 - x), \frac{1}{2} \leqq x \leqq 1)$, we have

$$\begin{aligned} J_1(u) &= (1 + i\gamma) \int_0^{\frac{1}{2}} (e^{x(1+i\gamma)} - 1)^{iu+2}(e - e^{x(1+i\gamma)})^{-iu+2} f(x + i\gamma x)\, dx \\ &+ (1 - i\gamma) \int_{\frac{1}{2}}^1 (e^{x+i\gamma(1-x)} - 1)^{iu+2}(e - e^{x+i\gamma(1-x)})^{-iu+2} f(x + i\gamma(1 - x))dx \end{aligned} \tag{2.2}$$

Clearly

$$\begin{aligned} am\left(\frac{e^{x(1+i\gamma)} - 1}{e - e^{x(1+i\gamma)}}\right) &= \tan^{-1} \frac{\sin \gamma x}{\cos \gamma x - e^{-x}} + \tan^{-1} \frac{\sin \gamma x}{e^{1-x} - \cos \gamma x} \\ &\geqq \tan^{-1} \frac{\sin \gamma x}{\cos \gamma x - e^{-x}}. \end{aligned}$$

But for $0 \leqq x \leqq \frac{1}{2}$, $x \geqq \cos \gamma x - e^{-x}$ since this is true for $x = 0$ and $\frac{d}{dx}(x - \cos \gamma x + e^{-x}) = 1 + \gamma \sin \gamma x - e^{-x} \geqq 0$ for $0 < \gamma < \frac{1}{2}\pi$, $0 \leqq x \leqq \frac{1}{2}$. Thus, for $0 \leqq x \leqq \frac{1}{2}$

$$\begin{aligned} am\left(\frac{e^{x(1+i\gamma)} - 1}{1 - e^{x(1+i\gamma)}}\right) &\geqq \tan^{-1} \frac{\sin \gamma x}{x} \geqq \tan^{-1}(2 \sin \tfrac{1}{2}\gamma) \\ &\geqq \tan^{-1}(2 \cos \tfrac{1}{2}\gamma \tan \tfrac{1}{2}\gamma) \\ &> \tan^{-1}(\tan \tfrac{1}{2}\gamma) = \tfrac{1}{2}\gamma. \end{aligned}$$

Thus, for $u > 0$

$$\begin{aligned} &\left|(1 + i\gamma) \int_0^{\frac{1}{2}} \left(\frac{e^{x(1+i\gamma)} - 1}{e - e^{x(1+i\gamma)}}\right)^{iu} (e^{x+i\gamma x} - 1)^2(e - e^{x+i\gamma x})^2 f(x + i\gamma x)\, dx\right| \\ &\qquad \leqq \tfrac{1}{2}\pi e^{-\frac{1}{2}\gamma u}(2e)^4 \operatorname{Max} |f(z)| \\ &\qquad \leqq 2^{13} e^{-\frac{1}{2}\gamma u} \operatorname{Max} |f(z)|. \end{aligned}$$

In precisely the same way this holds for the second integral on the right of (2.2). This completes the lemma.

Lemma 3. *Let*

$$J_2(u, v) = \int_0^1 (e^x - 1)^{iu+2}(e - e^x)^{-iu+2}e^{-ivx}\,dx. \tag{2.3}$$

Then for $u > 0$

$$|J_2(u, v)| < 10^5 e^{-\frac{1}{2}\gamma(u-v)}, \qquad v > 0, \tag{2.4}$$

$$|J_2(u, v)| < 10^5 e^{-\frac{1}{2}\gamma u}, \qquad v < 0. \tag{2.5}$$

Also there exists a constant C *such that*

$$|J_2(u, v)| < C\frac{1 + u^2}{1 + v^2}. \tag{2.6}$$

Proof. If in Lemma 2 we replace $f(x)$ by e^{-ivx} then $|e^{-iv(x+iy)}| = e^{vy}$, $0 \leqq y \leqq \frac{1}{2}\gamma$. Thus for $v > 0$, $|e^{-iv(x+iy)}| \leqq e^{\frac{1}{2}v\gamma}$ and for $v < 0$, $|e^{-iv(x+iy)}| \leqq 1$. Thus (2.4) and (2.5) follow from Lemma 2.

(2.6) is easily obtained from (2.3) by two integrations by parts.

3. In proving Theorem I we consider only the case $a = 0$, $c = 1$. (Then $0 < b < 1$.) It is quite trivial to show that all other cases are reducible to this one.

Proof of Theorem I. Part a. Let

$$h(w) = \int_b^1 \{F(x) - f(x)\}(e^x - 1)^{2+iw}(e - e^x)^{2-iw}\,dx.$$

If $w = u + iv$ then for $v > 0$, $h(w)$ is analytic, and for $v \geqq 0$, $h(w)$ is continuous. Also

$$|h(w)| \leqq (e^b - 1)^{-v}(e - e^b)^v e^4 \int_b^1 |F(x) - f(x)|\,dx.$$

If β is defined by $(e - e^b)/(e^b - 1) = e^\beta$ then

$$|h(w)| \leqq e^{\beta v+4}\int_b^1 |F(x) - f(x)|\,dx < Ae^{\beta v}, \tag{3.0}$$

where A is used to represent any constant that depends on $F(x)$ and $f(x)$.

By (2.0) it is clear that

$$h(u) = -J_1(u) + \int_0^1 F(x)(e^x - 1)^{2+iu}(e - e^x)^{2-iu}\,dx.$$

But using (2.3)

$$\int_0^1 F(x)(e^x - 1)^{2+iu}(e - e^x)^{2-iu}\,dx$$

$$= \int_0^1 (e^x - 1)^{2+iu}(e - e^x)^{2-iu}\,dx\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} G(y)e^{-iyx}\,dy.$$

Thus

$$h(u) = -J_1(u) + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} G(y)J_2(u, y)\,dy. \tag{3.1}$$

Clearly

$$\left|\int_{-\infty}^{\infty} G(y)\,J_1(u, y)\,dy\right| \leqq \int_{-\infty}^{\frac{1}{2}u} |\,G(y)\,J_2(u, y)\,|\,dy + \int_{\frac{1}{2}u}^{\infty} |\,G(y)J_2(u, y)\,|\,dy$$

and using Lemma 3, this gives for $u > 1$

$$\left|\int_{-\infty}^{\infty} G(y)\,J_2(u, y)\,dy\right| \leqq Ae^{-\frac{1}{2}\gamma u} + Au^2\int_{\frac{1}{2}u}^{\infty}\frac{G(y)}{y^2}\,dy.$$

Thus by (1.1), (2.1), and (3.1)

$$|\,h(u)\,| \leqq Au^2(e^{-\frac{1}{2}\gamma u} + e^{-\theta(\frac{1}{2}u)}), \qquad u > 1.$$

This last inequality, (1.2), and Lemma 1 give

$$\left(\int_{-\infty}^{-1} + \int_1^{\infty}\right)\frac{\log^-|\,h(u)\,|}{u^2}\,du = -\infty. \tag{3.2}$$

Part b. Let us assume that $h(w)$ is not identically zero. Then by using Carleman's theorem[4] on $h(w)$ in the upper half-plane

$$0 < \frac{1}{2\pi}\int_1^R \log|\,h(u)\,h(-u)\,|\left(\frac{1}{u^2} - \frac{1}{R^2}\right)du$$

$$+ \frac{1}{\pi R}\int_0^{\pi} \log|\,h(Re^{i\theta})\,|\sin\theta\,d\theta + A.$$

[4] Titchmarsh, Theory of Functions, Oxford, 1932, p. 130.

Or

$$-\left(\int_{-R}^{-1}+\int_{1}^{R}\right)\log^{-}|h(u)|\left(\frac{1}{u^2}-\frac{1}{R^2}\right)du$$
$$\leqq\int_{1}^{R}\frac{\log^{+}|h(u)|+\log^{+}|h(-u)|}{u^2}\,du$$
$$+\frac{2}{R}\int_{0}^{\pi}\log^{+}|h(Re^{i\theta})|\sin\theta\,d\theta+A.$$

Using (3.0) this gives

$$-\left(\int_{-kR}^{-1}+\int_{1}^{kR}\log^{-}|h(u)|\left(\frac{1}{u^2}-\frac{1}{R^2}\right)du<A.\right.$$

From this

$$-\left(\int_{-kR}^{-1}+\int_{1}^{kR}\right)\log^{-}|h(u)|\frac{du}{2u^2}<A.$$

Thus

$$\left(\int_{-kR}^{-1}+\int_{1}^{kR}\right)\frac{\log^{-}|h(u)|}{u^2}\,du>-A.$$

Since A is independent of R, this contradicts (3.2). Thus $h(w)=0$, and therefore

$$(3.3)\qquad\int_{0}^{1}|F(x)-f(x)|\left(\frac{e^x-1}{e-e^x}\right)^{iu}(e^x-1)^2(e-e^x)^2\,dx=0.$$

Part c. If we now set $e^t=(e^x-1)/(e-e^x)$ then the interval $0<x<1$ maps over to the interval $-\infty<t<\infty$ and (3.3) becomes the Fourier transform of an integrable function. But only a function equivalent to zero can have a transform that vanishes identically. This means that the function obtained by mapping $F(x)-f(x)$ on the interval $-\infty<t<\infty$ vanishes almost everywhere. Thus $F(x)-f(x)$ vanishes almost everywhere on (0, 1). But this proves our theorem.

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A THEOREM ON ABSOLUTELY CONVERGENT TRIGONOMETRICAL SERIES

By H. R. Pitt

(1.1) The object of this paper is to extend to general trigonometrical series Wiener's[1] well known theorem that if $f(x)$ is periodic and has an absolutely convergent Fourier series, and if $f(x)$ does not vanish, then $[f(x)]^{-1}$ also has an absolutely convergent Fourier series.

The theorem we are concerned with was proved independently by R. H. Cameron[2] and myself. Cameron's theorem is more general in that he considers functions of infinitely many variables, but it seems worth while to give a more elementary proof for the case of one variable.

We denote by $\mathfrak{A}$ the class of functions $f(x)$ for which we can write

$$f(x) = \sum_{n=1}^{\infty} a_n e^{i\lambda_n x}, \qquad T\{f(x)\} = \sum |a_n| < \infty.$$

(2) Theorem. *If $f(x)$ belongs to $\mathfrak{A}$, and if the lower bound of $|f(x)|$ in $(-\infty, \infty)$ is positive, then $[f(x)]^{-1}$ belongs to $\mathfrak{A}$.*

We require three elementary lemmas.

(2.1) Lemma 1. *If $f_1(x)$, $f_2(x)$ belong to $\mathfrak{A}$, then so do $f_1(x)f_2(x)$ and $c_1f_1(x) + c_2f_2(x)$, c_1 and c_2 being any real or complex numbers. Moreover,*

(a) $$T\{c_1f_1(x) + c_2f_2(x)\} \leqq |c_1|T\{f_1(x)\} + |c_2|T\{f_2(x)\},$$

(b) $$T\{f_1(x)f_2(x)\} \leqq T\{f_1(x)\} \cdot T\{f_2(x)\}.$$

The first result is obvious. To prove the second, we write

$$f_1(x) = \sum a_n e^{-i\rho_n x}, \qquad f_2(x) = \sum b_n e^{-i\tau_n x}.$$

Then

$$f_1(x)f_2(x) = \sum_k e^{-i\lambda_k x} \sum_{\rho_n+\tau_m=\lambda_k} a_n b_m,$$

$$T\{f_1(x)f_2(x)\} = \sum \Big|\sum a_n b_m\Big|$$

$$\leqq \sum |a_n| \sum |b_m| = T\{f_1(x)\} \cdot T\{f_2(x)\}.$$

[1] N. Wiener, *Tauberian theorems*, Annals of Mathematics, 33 (1932), 1–100.

[2] R. H. Cameron, *Analytic functions of general trigonometrical sums*, Duke Math. Journ., 3 (1937).

(2.2) LEMMA 2. *If $f(x)$ belongs to $\mathfrak{A}$ and $T\{f(x)\} = \alpha < 1$, then $[1 + f(x)]^{-1}$ belongs to $\mathfrak{A}$ and*

$$T\{[1 + f(x)]^{-1}\} \leqq (1 - \alpha)^{-1}.$$

We have

$$[1 + f(x)]^{-1} = \sum_{n=0}^{\infty} (-1)^n [f(x)]^n, \tag{2.2.1}$$

the series being absolutely and uniformly convergent for $-\infty < x < \infty$. It follows from Lemma 1 (b) that $[f(x)]^n$ belongs to $\mathfrak{A}$ and that $T\{[f(x)]^n\} \leqq \alpha^n$. If we write each term of (2.2.1) as a trigonometrical series, it becomes clear, since $\alpha < 1$, that $[1 + f(x)]^{-1}$ belongs to $\mathfrak{A}$ and that

$$T\{[1 + f(x)]^{-1}\} \leqq \sum_{n=0}^{\infty} \alpha^n = (1 - \alpha)^{-1}.$$

(2.3) Let $\epsilon > 0$, $\lambda > 0$, and suppose that

$$\frac{2\pi}{3\lambda\epsilon} = K$$

is an integer greater than 1. Let

$$\omega(\epsilon, \lambda, x) = 1 \qquad (0 \leqq |x| < \epsilon),$$

$$\omega(\epsilon, \lambda, x) = 2 - |x|/\epsilon \qquad (\epsilon < |x| \leqq 2\epsilon),$$

$$\omega(\epsilon, \lambda, x) = 0 \qquad (2\epsilon < |x| \leqq \pi/\lambda),$$

$$\omega(\epsilon, \lambda, x + 2\pi/\lambda) = \omega(\epsilon, \lambda, x).$$

LEMMA 3 (a) *$\omega(\epsilon, \lambda, x)$ belongs to $\mathfrak{A}$, and*

$$T\{\omega(\epsilon, \lambda, x)\} \leqq C,$$

where C is independent of ϵ and λ.
(b) *For fixed λ,*

$$\lim_{K \to \infty} T\{\omega(\epsilon, \lambda, x)[1 - e^{-i\lambda x}]\} = 0.$$

An easy calculation shows that

$$\omega(\epsilon, \lambda, x) = \sum_{-\infty}^{\infty} d_n e^{-i\lambda n x},$$

where

$$d_0 = \frac{3\lambda\epsilon}{2\pi}, \qquad d_n = \frac{2}{\pi\lambda\epsilon n^2} \sin\frac{n\lambda\epsilon}{2} \sin\frac{3n\lambda\epsilon}{2} \qquad (n \neq 0).$$

It is plain that $\omega(\epsilon, \lambda, x)$ belongs to $\mathfrak{A}$. Further,

$$T\{\omega(\epsilon, \lambda, x)\} \leqq \sum_{3\lambda\epsilon|n| \leqq \pi} |d_n| + \frac{2}{\pi\lambda\epsilon} \sum_{3\lambda\epsilon|n|>\pi} \frac{1}{n^2}$$

$$\leqq \frac{3\lambda\epsilon}{2\pi} + 1 + \frac{2}{\pi\lambda\epsilon} \sum_{3\lambda\epsilon|n|>\pi} \frac{1}{n^2} \leqq C,$$

since

$$2\pi/3\lambda\epsilon = K \geqq 2.$$

(b) $$\omega(\epsilon, \lambda, x)[1 - e^{-i\lambda x}] = \sum (d_n - d_{n-1})e^{-i\lambda n x}$$

It is easy to show that $\lim_{K\to\infty} (d_n - d_{n-1}) = 0$ for each n and that

$$|d_n - d_{n-1}| \leqq A/n^2 \qquad (n \neq 0, n \neq 1),$$

where A is independent of K. The conclusion follows by uniform convergence.

(2.4) We can now complete the proof of the theorem. Let

(2.4.1) $$f(x) = \sum_{n=1}^{\infty} a_n e^{-i\lambda_n x},$$

(2.4.2) $$|f(x)| \geqq \delta > 0 \qquad (-\infty < x < \infty).$$

Then

$$f(x) = f_1(x) + f_2(x),$$

where

$$f_1(x) = \sum_{n=1}^{N} a_n e^{-i\lambda_n x}, \qquad f_2(x) = \sum_{n=N+1}^{\infty} a_n e^{-i\lambda_n x},$$

and N is chosen so that

(2.4.3) $$T\{f_2(x)\} \leqq \tfrac{1}{4}\delta,$$

(2.4.4) $$|f_1(x)| \geqq \tfrac{3}{4}\delta \qquad (-\infty < x < \infty).$$

Let K be an integer greater than 1 and let

(2.4.5) $$\epsilon_n = \frac{2\pi}{3K\lambda_n}, \qquad \omega_n(x) = \omega(\epsilon_n, \lambda_n, x)$$

for $n = 1, 2, \cdots N$. Then

$$\sum_{k=1}^{K} \omega_n(x - 3k\epsilon_n) = 1$$

for $n = 1, 2, \cdots N$ and all x, and it follows that

$$\prod_{n=1}^{N} \sum_{k=1}^{K} \omega_n(x - 3k\epsilon_n) = 1$$

Hence

$$\sum \prod_{n=1}^{N} \omega_n(x - 3k_n\epsilon_n) = 1, \tag{2.4.6}$$

where the summation extends over the K^N rearrangements $(k_1, \cdots, k_n)$ of the integers $(1, 2, \cdots K)$. We can therefore write

$$\frac{1}{f(x)} = \sum \frac{\prod_{n=1}^{N} \omega_n(x - 3\epsilon_n k_n)}{f(x)}. \tag{2.4.7}$$

By Lemma 1, it is sufficient to prove that each term of this sum belongs to $\mathfrak{A}$.

Let x_0 be a point at which

$$\Omega(x) = \prod_{n=1}^{N} \omega_n(x - 3\epsilon_n k_n) \neq 0.$$

(If there is no such point, the relevant term of (2.4.7) vanishes identically.) Then

$$\frac{\Omega(x)}{f(x)} = \frac{\Omega(x)}{f_1(x_0) + f_2(x) + [f_1(x) - f_1(x_0)] \prod_{n=1}^{N} \omega(2\epsilon_n, \lambda_n, x - 3\epsilon_n k_n)}, \tag{2.4.8}$$

since the denominators differ only at points where $\Omega(x)$ vanishes. Since

$$[f_1(x) - f_1(x_0)] \prod_{n=1}^{N} \omega(2\epsilon_n, \lambda_n, x - 3\epsilon_n k_n)$$

$$= \sum_{\nu=1}^{N} a_\nu (e^{-\Omega_\nu x} - e^{-\Omega_\nu x_0}) \prod_{n=1}^{N} \omega(2\epsilon_n, \lambda_n, x - 3\epsilon_n k_n),$$

we have

$$T\left\{[f_1(x) - f_1(x_0)] \prod_{n=1}^{N} \omega(2\epsilon_n, \lambda_n, x - 3\epsilon_n k_n)\right\}$$

$$\leqq \sum_{\nu=1}^{N} |a_\nu| \, C^{N-1} T\{[e^{-\Omega_\nu x} - e^{-\Omega_\nu x_0}]\omega(2\epsilon_\nu, \lambda_\nu, x - 3\epsilon_\nu k_\nu)\}, \tag{2.4.9}$$

by Lemmas 1 and 3. Since $\Omega(x_0) \neq 0$, we have

$$\omega_\nu(x_0 - 3\epsilon_\nu k_\nu) \neq 0 \qquad (\nu = 1, 2, \cdots N),$$

and therefore, by the definition of $\omega_\nu(x)$,

$$|x_0 - 3\epsilon_\nu k_\nu| < 2\epsilon_\nu, \qquad \text{mod } (2\pi/\lambda_\nu), \tag{2.4.10}$$

for $\nu = 1, 2, \cdots N$. Returning to (2.4.9), we see that

$$\begin{aligned} T\{[e^{-\Omega_\nu x} - e^{-\Omega_\nu x_0}]\omega(2\epsilon_\nu, \lambda_\nu, x - 3\epsilon_\nu k_\nu)\} \\ &= T\{[e^{-\Omega_\nu(y+3\epsilon_\nu k_\nu)} - e^{-\Omega_\nu x_0}]\omega(2\epsilon_\nu, \lambda_\nu, y)\} \\ &= T\{[e^{-\Omega_\nu y} - e^{-\Omega_\nu(x_0 - 3\epsilon_\nu k_\nu)}]\omega(2\epsilon_\nu, \lambda_\nu, y)\} \\ &\leqq T\{[1 - e^{-\Omega_\nu y}]\omega(2\epsilon_\nu, \lambda_\nu, y)\} + C\,|\,1 - e^{-\Omega_\nu(x_0 - 3\epsilon_\nu k_\nu)}\,|, \end{aligned}$$

by Lemma 3 (a). It follows now from (2.4.5), (2.4.10) and Lemma 3 (b) that the last expression tends to zero, as $K \to \infty$, for $\nu = 1, 2, \cdots N$. Hence, by (2.4.3), (2.4.9) and Lemma 1 (a),

$$T\left\{f_2(x) + [f_1(x) - f_1(x_0)] \prod_{n=1}^{N} \omega(2\epsilon_n, \lambda_n, x - 3\epsilon_n k_n)\right\} < \tfrac{3}{4}\delta$$

if K is sufficiently large, and therefore

$$\left[f_1(x_0) + f_2(x) + [f_1(x) - f_1(x_0)] \prod_{n=1}^{N} \omega(2\epsilon_n, \lambda_n, x - 3\epsilon_n k_n)\right]^{-1}$$

belongs to $\mathfrak{A}$, by Lemma 2. The conclusion follows from (2.4.7), (2.4.8) and Lemma 1.

(2.5) In conclusion, we may remark that the method of proof which we have used here can be adapted without difficulty to prove the following result:

If $f(x)$ belongs to $\mathfrak{A}$, and if its values lie in a domain in which $\Phi(z)$ is a regular analytic function, then $\Phi[f(x)]$ belongs to $\mathfrak{A}$. This extension, in the case of Wiener's theorem, was pointed out by P. Lévy.[3]

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[3] P. Lévy, *Sur le convergence absolue des séries de Fourier*, Comptes Rendues, 196 (1933), 463.

SOLUTION OF LAPLACE'S HARMONIC EQUATION WITH THREE INDEPENDENT VARIABLES

By S. A. Savin

1. Three compound arguments

In order to integrate Laplace's harmonic equation

$$\nabla^2 u(x, y, z) \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{1}$$

let us take the arbitrary function

$$u = f(k_0, k, r) \tag{2}$$

with three compound arguments k_0, k, r defined by the formulas

$$k_0 = \alpha_0 x + \epsilon_0 y + \nu_0 z, \qquad k = \alpha x + \epsilon y + \nu z, \tag{3}$$

$$r = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} \tag{4}$$

where

$$\alpha_0, \epsilon_0, \nu_0; \qquad \alpha, \epsilon, \nu; \qquad a, b, c$$

are arbitrary constants.

We subordinate these arbitrary constants only to one condition that the three compound arguments be independent of one another.

2. Condition of the independence of three compound arguments

For obtaining more compact formulas let us use the negative exponents of degree instead of fractions.

The Jacobian of the three compound arguments k_0, k, r is equal to

$$J = \begin{vmatrix} \alpha_0 & \epsilon_0 & \nu_0 \\ \alpha & \epsilon & \nu \\ \dfrac{x-a}{r} & \dfrac{y-b}{r} & \dfrac{z-c}{r} \end{vmatrix}$$

or in the developed form

$$J = r^{-1}[(x-a)(\epsilon_0\nu - \nu_0\epsilon) + (y-b)(\nu_0\alpha - \alpha_0\nu) + (z-c)(\alpha_0\epsilon - \epsilon_0\alpha)].$$

It may be equal to zero only on condition

$$\frac{\alpha_0}{\alpha} = \frac{\epsilon_0}{\epsilon} = \frac{\nu_0}{\nu}$$

as x, y, z are independent variables.

Hence we obtain the condition of the independence of the three compound arguments k_0, k, r:

$$\frac{\alpha_0}{\alpha} \neq \frac{\epsilon_0}{\epsilon} \neq \frac{\nu_0}{\nu} \tag{5}$$

i.e. all the three corresponding coefficients in the expressions k_0 and k are not to be proportional.

3. Formation of the Laplacian of the function (2)

Let us form from the function (2) partial derivatives of the first order

$$\frac{\partial u}{\partial x} = \alpha_0 \frac{\partial f}{\partial k_0} + \alpha \frac{\partial f}{\partial k} + (x - a) r^{-1} \cdot \frac{\partial f}{\partial r}$$

and then of the second order

$$\frac{\partial^2 u}{\partial x^2} = \alpha_0^2 \frac{\partial^2 f}{\partial k_0^2} + \alpha^2 \frac{\partial^2 f}{\partial k^2} + (x - a)^2 r^{-2} \cdot \frac{\partial^2 f}{\partial r^2} + 2\alpha_0 \alpha \frac{\partial^2 f}{\partial k_0 \partial k}$$
$$+ 2(x - a) r^{-1} \cdot \left[\alpha_0 \frac{\partial^2 f}{\partial k_0 \partial r} + \alpha \frac{\partial^2 f}{\partial k \partial r} \right] + r^{-1} \cdot \left[1 - (x - a)^2 r^{-2} \right] \frac{\partial f}{\partial r}.$$

Hence, in consequence of the symmetry of formulas—two other partial derivatives of the second order

$$\frac{\partial^2 u}{\partial y^2}, \qquad \frac{\partial^2 u}{\partial z^2}$$

may be obtained by cyclic permutation of four cycles of letters:

$$(x, y, z), \quad (\alpha_0, \epsilon_0, \nu_0), \quad (\alpha, \epsilon, \nu), \quad (a, b, c).$$

The addition of these three partial derivatives of the second order gives—according to (3) and (4), the following expression of the Laplacian of the function (2):

$$\nabla^2 u = (\alpha_0^2 + \epsilon_0^2 + \nu_0^2) \frac{\partial^2 f}{\partial k_0^2} + (\alpha^2 + \epsilon^2 + \nu^2) \frac{\partial^2 f}{\partial k^2} + \frac{\partial^2 f}{\partial r^2}$$
$$+ 2(\alpha_0 \alpha + \epsilon_0 \epsilon + \nu_0 \nu) \frac{\partial^2 f}{\partial k_0 \partial k} + 2r^{-1} \cdot \left[(k_0 + h_0) \frac{\partial^2 f}{\partial k_0 \partial r} \right.$$
$$\left. + (k + h) \frac{\partial^2 f}{\partial k \partial r} + \frac{\partial f}{\partial r} \right],$$

where, for brevity, the new constants are introduced:

$$h_0 = -(\alpha_0 a + \epsilon_0 b + \nu_0 c), \qquad h = -(\alpha a + \epsilon b + \nu c). \tag{6}$$

Taking into consideration the relation

$$\frac{\partial^2 f}{\partial r^2} + 2r^{-1}\cdot\left[(k_0 + h_0)\frac{\partial^2 f}{\partial k_0 \partial r} + (k + h)\frac{\partial^2 f}{\partial k \partial r} + \frac{\partial f}{\partial r}\right]$$
$$= r^{-1}\cdot\frac{\partial}{\partial r}\left[2(k_0 + h_0)\frac{\partial f}{\partial k_0} + 2(k + h)\frac{\partial f}{\partial k} + r\frac{\partial f}{\partial r} + f\right]$$

we simplify the obtained expression

$$\nabla^2 u = (\alpha_0^2 + \epsilon_0^2 + \nu_0^2)\frac{\partial^2 f}{\partial k_0^2} + (\alpha^2 + \epsilon^2 + \nu^2)\frac{\partial^2 f}{\partial k^2}$$
$$+ 2(\alpha_0\alpha + \epsilon_0\epsilon + \nu_0\nu)\frac{\partial^2 f}{\partial k_0 \partial k} + r^{-1}\cdot\frac{\partial}{\partial r}\left[2(k_0 + h_0)\frac{\partial f}{\partial k_0}\right. \tag{7}$$
$$\left. + 2(k + h)\frac{\partial f}{\partial k} + r\frac{\partial f}{\partial r} + f\right].$$

4. Integrating of an auxiliary equation

According to the initial condition the function f is arbitrary, the arguments k_0, k are independent of one another—which is expressed by the inequalities (5). Therefore in order to annul the obtained expression of the Laplacian (7) it is sufficient to have four compatible conditions:

$$\frac{\partial^2 f}{\partial k_0^2} = 0, \tag{8}$$

$$\alpha^2 + \epsilon^2 + \nu^2 = 0, \qquad \alpha_0\alpha + \epsilon_0\epsilon + \nu_0\nu = 0, \tag{9}$$

$$\frac{\partial}{\partial r}\left[2(k_0 + h_0)\frac{\partial f}{\partial k_0} + 2(k + h)\frac{\partial f}{\partial k} + r\frac{\partial f}{\partial r} + f\right] = 0. \tag{10}$$

Considering the last condition let us at first integrate the auxiliary equation

$$2(k_0 + h_0)\frac{\partial f}{\partial k_0} + 2(k + h)\frac{\partial f}{\partial k} + r\frac{\partial f}{\partial r} = -f. \tag{11}$$

To this equation corresponds the system

$$\frac{dk_0}{2(k_0 + h_0)} = \frac{dk}{2(k + h)} = \frac{dr}{r} = \frac{df}{-f},$$

where according to (6), h_0 and h—are constants. By integrating this system we obtain:

$$(k_0 + h_0)\cdot f^2 = C_1\,, \qquad (k + h)\cdot f^2 = C_2\,, \qquad rf = C_3\,,$$

where C_1, C_2, C_3—are the constants of integration. Let us transform these three equalities into the following:

$$m_0 = (k_0 + h_0)r^{-2}, \qquad m = (k + h)r^{-2}, \tag{12}$$
$$m_1 = \sqrt{Ar^2 + B(k + h) + D(k_0 + h_0)}\cdot f,$$

where A, B, D are arbitrary constants and m_0, m, m_1—the new arguments.

Hence we obtain the general integral of the equation (11):

$$f = [Ar^2 + B(k + h) + D(k_0 + h_0)]^{-\frac{1}{2}}\cdot\varphi_0(m_0\,, m), \tag{13}$$

where φ_0 is an arbitrary function.

The expression (13) is a particular integral for the equation (10). By adding to it the arbitrary function $\psi_0(k_0\,, k)$ we obtain the integral of the equation (10) in more general form:

$$f = [Ar^2 + B(k + h) + D(k_0 + h_0)]^{-\frac{1}{2}}\cdot\varphi_0(m_0\,, m) + \psi_0(k_0\,, k). \tag{14}$$

5. Formula of the integral of Laplace's equation (1)

In order that the function (14) also be an integral of Laplace's harmonic equation—it is to be subjected to the conditions (8) and (9).

Supposing

$$D = 0,$$
$$\varphi_0(m_0, m) = m_0\varphi_1(m) + \varphi_2(m),$$
$$\psi_0(k_0, k) = k_0\psi_1(k) + \psi_2(k),$$

we shall obtain the function f which satisfies the condition (8).

Further, let us subordinate the coefficients of the linear arguments k_0, k to the conditions (9).

Introducing these values into the formula (14) we obtain according to (2) the solution of the equation (1) in the following abridged form:

$$u = (C_0 + k_0)\psi(k) + \frac{C + m_0}{\sqrt{Ar^2 + B(k + h)}}\,\varphi(m) \tag{15}$$

where A, B, C, C_0—are arbitrary constants, and ψ, φ—arbitrary functions. Here the arguments k_0, k are defined by the formulas (3) and

are connected by the relations (9). The argument r is calculated by (4). The compound arguments m_0 and m, newly introduced, are defined by the formulas (12) according to the denotations (3), (4), (6) and condition (5).

On condition

$$A = 1, \qquad B = 0, \qquad \alpha_0 = \epsilon_0 = \nu_0 = 0$$

the expression (15) gives a simple example of the property of a harmonic function, stated in 1845 by Lord Kelvin, namely—that if $U(x, y, z)$ represents a harmonic function—the function

$$\frac{1}{r} U\left(\frac{x}{r^2}, \frac{y}{r^2}, \frac{z}{r^2}\right)$$

is also harmonic, i.e.—it satisfies Laplace's equation (Goursat, E. Cours d'analyse mathématique, t. 3, Paris, 1923, p. 243).

6. About one-valued dependence of the variables x, y, z from compound arguments k_0, k, r

By means of the formulas (3), (4) and relations (9) it is easy to prove that between the two systems: of the variables x, y, z and the arguments k_0, k, r—there is an one-one-correspondence.

In order to prove the above let us determine at first the variables x and y as functions of the arguments k_0, k, r.

According to the denotations (3) and (6) we form the system

$$\alpha_0(x - a) + \epsilon_0(y - b) = k_0 + h_0 - \nu_0(z - c),$$

$$\alpha (x - a) + \epsilon (y - b) = k + h - \nu (z - c).$$

Solving it we find:

$$x - a = (\alpha\epsilon_0 - \epsilon\alpha_0)^{-1} \cdot [(\epsilon\nu_0 - \nu\epsilon_0)(z - c) - \epsilon(k_0 + h_0) + \epsilon_0(k + h)],$$

$$y - b = (\alpha\epsilon_0 - \epsilon\alpha_0)^{-1} \cdot [(\nu\alpha_0 - \alpha\nu_0)(z - c) - \alpha_0(k + h) + \alpha(k_0 + h_0)].$$

Inserting these values into (4) we obtain a quadratic equation in $(z - c)$. The coefficient of $(z - c)^2$ in this equation is equal to

$$(\alpha\epsilon_0 - \epsilon\alpha_0)^2 + (\epsilon\nu_0 - \nu\epsilon_0)^2 + (\nu\alpha_0 - \alpha\nu_0)^2.$$

According to the well known identity of Euler-Lagrange and to the dependences (9), we have:

$$(\alpha\epsilon_0 - \epsilon\alpha_0)^2 + (\epsilon\nu_0 - \nu\epsilon_0)^2 + (\nu\alpha_0 - \alpha\nu_0)^2$$
$$= (\alpha^2 + \epsilon^2 + \nu^2)(\alpha^2 + \epsilon^2 + \nu^2) - (\alpha\alpha_0 + \epsilon\epsilon_0 + \nu\nu_0) = 0.$$

This shows that $(z - c)$ is also defined by the linear equation as well as $(x - a)$, $(y - b)$, i.e. x, y, z—become one-valued functions from the arguments k_0, k, r.

7. On the formation of coefficients for the compound arguments k_0, k

In 1902 E. T. Whittaker, Sc.D., Professor of Mathematics, University of Edinburgh, suggested solving Laplace's harmonic equation (1) by means of a definite integral, the integrand of which contains a trinomial with the coefficients:

$$1,\ i\cos t,\ i\sin t, \qquad \text{or} \qquad \cos t,\ \sin t,\ i,\ (i = \sqrt{-1})$$

(Whittaker and Watson, A Course of Modern Analysis, Cambridge, 1920, p. 388–390).

The coefficients of that type may also be used for forming the arguments k_0 and k, as they are connected by the relations (5) and (9).

In fact, the equation $I(9)$ gives two solutions and shows that the coefficients α, ϵ, ν—are complex quantities.

Suppose that the coefficients of the argument k are equal to:

$$\alpha = \cos\theta, \qquad \epsilon = \pm i, \qquad \nu = \sin\theta,$$

which satisfies the condition $I(9)$. Then let us take the coefficients of the argument (k_0) in the following symmetrical form:

$$\alpha_0 = B\nu - \epsilon C, \qquad \epsilon_0 = C\alpha - \nu A, \qquad \nu_0 = A\epsilon - \alpha B.$$

Evidently, these coefficients satisfy identically the condition $II(9)$.

In these expressions of the coefficients—the values θ, A, B, C—are arbitrary constants. They must be chosen thus as to satisfy the condition (5) which is necessary for the independence of one another of the three compound arguments k_0, k, r.

Supposing in these formulas

$$\theta = 0, \qquad A = C = 0, \qquad B = -1$$

we obtain a particular case with a simpler form of arguments:

$$k_0 = z, \qquad k = x \pm iy.$$

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ON THE RELATION BETWEEN SIMPLE DISTILLATION AND TRUE BOILING POINT CURVES OF COMPLEX MIXTURES THE COMPONENTS OF WHICH OBEY RAOULT'S LAW

By E. M. J. Pease

Let the composition of a complex liquid mixture (for example, of hydrocarbons) be represented by a "true boiling-point curve", in which cumulative amount x is plotted against temperature T. (Amounts will be expressed through-out in mols, on the assumption that information is available for conversion, if later desired, from mol % to weight % or volume %). Physically, such a curve indicates that the amount dx of the liquid has a true boiling point range of T to $T + dT$. Let the total amount initially be unity.

Let the mixture be subjected to a simple distillation i.e., to one in which the vapor leaving the liquid, with which it is in equilibrium, is completely removed from the system as fast as formed, and there is no reflux. At any time during the course of the distillation, when the amount Y has been collected in the receiver and $1 - Y$ is left in the still, let the true boiling point curve of the residue in the still be represented by the curve T vs. z, beginning and ending at the same temperatures as the $T - x$ curve, and having a value of z varying from $1 - Y$ to 0. For each quantity dx_1, in the original mixture and having a boiling range T_1 to $T_1 + dT_1$, there is a corresponding quantity dz_1, in the mixture in the still, which has the same boiling range. Consider a component of the mixture, in original amount dx_n, which boils in the neighborhood of T_n, and let this be called the key component (practical limitations on its choice will be considered later.)

With the amount $1 - Y$ left in the still, distill off the amount dY containing $d(dz_1)$ of the components boiling near T_1 and $d(dz_n)$ of the key component boiling near T_n. The ratio in the vapor of the amounts of the two components under consideration, is given by the relation

$$\frac{d(dz_1)}{d(dz_n)} = \frac{dz_1}{dz_n} \cdot \alpha_1$$

in which $d(dz_1)/d(dz_n)$ is their ratio in the vapor, dz_1/dz_n is their ratio in the liquid, and α_1 is the volatility of a component boiling at T_1 relative to the volatility of the key component boiling at T_n. For mixtures of

compounds in which Raoult's law is applicable, α is the ratio of the vapor pressures of the pure components under consideration, at the temperature of the distillation. Although this ratio depends somewhat on temperature, the assumption that it is constant will not introduce large error. On making the assumption of constancy of α, equation (1) yields

$$\int_{dz_1}^{dz_1} d(dz_1)/dz_1 = \alpha_1 \int_{dz_n}^{dz_n} d(dz_n)/dz_n$$

or

$$dz_1 = dx_1(dz_n/dx_n)\alpha_1 \tag{2}$$

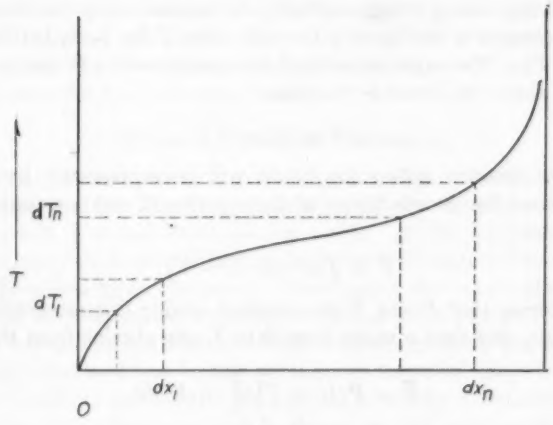

FIG. 1

The ratio dz_n/dx_n is the fractional part of the key component boiling at T_n, which is left in the still when the part Y of the original mixture has deen distilled off. Call this ratio F. Subscripts may now be dropped, and the general relation between the amount dx in the original mixture which has the boiling range T to $T + dT$, and the corresponding amount dz in the still which has the same boiling point range is

$$dz = F^{\alpha} dx \tag{2a}$$

in which α depends on x and on the choice of key-component. From (2a),

$$\int dz = \int F^{\alpha} dx$$

$$1 - Y = \int_0^1 F^{\alpha} dx \tag{3}$$

Equation (3) gives the relation between the part Y of the initial mixture which is distilled over by simple distillation and the fractional amount F, of some key-component, which is found remaining in the still. The temperature at this point in the distillation must also be determined. Let the temperature in the still be T when the amount Y has been distilled, and let the vapor pressure of the key-component at temperature T be represented by P. Of the component boiling between T and $T + dT$ and being present initially in amount dx in the liquid, the fraction present in the liquid in the still, when Y has been distilled off, is $dz/(1 - Y)$. The vapor pressure of this component is P, and its partial pressure above the liquid is therefore

$$\alpha P \cdot dz/(1 - Y)$$

The total pressure above the liquid will be represented by $\bar{P}$. Its value is fixed by the conditions of the experiment and is commonly one atmosphere.

$$\bar{P} = \int \alpha P \cdot dz/(1 - Y)$$

Remembering that P and Y are constant during this integration, that $dz = F^{\alpha} dx$, and that x varies from 0 to 1, one obtains from the above

$$\bar{P} = P/(1 - Y) \int_0^1 \alpha \cdot F^{\alpha} dx \tag{4}$$

It has already been pointed out that since x and α are both functions of T, $\alpha = f(x)$. Rewriting equations (3) and (4) together, with this substitution and in slightly different form, one has

$$Y = 1 - \int_0^1 F^{f(x)} dx \tag{3.1}$$

$$P = \frac{\bar{P} \int_0^1 F^{f(x)} dx}{\int_0^1 f(x) F^{f(x)} dx} \tag{4.1}$$

This, on inspection is an expression of the relation between Y and P in the form of a pair of parametric equations, with the parameter F

having any value between 0 and 1, each value determining one point on the $Y - P$ curve. If the true boiling-point curve (the $T - x$ relation) is given to start with, then the $\alpha - x$ relation, or form of the function $f(x)$, is given, and one point on the $Y - P$ curve is obtained for each value of F by straightforward graphical integration. Since the $P - T$ relation is available Y is therefore able to be determined as a function of T. This is the simple distillation curve of the mixture.

The more interesting and difficult case, the subject of the present mathematical investigation, is that in which the simple distillation curve, Y vs. T, is given. From the available $P - T$ relation, the $Y - P$ function is determined. The problem is, given the $Y - P$ relation and the condition that the above pair of parametric equations must be satisfied, to find the form of the function $f(x)$ expressing the relation between x and α. Since the value of α identifies a compound as to boiling point, the $x - T$ relation, or true boiling-point curve, follows directly from the $x - \alpha$ relation found.

Notes on Numerical Calculation

Given simple distillation curve of fractional part distilled over, Y, versus temperature T. The application of the equations discussed involves first a choice of key-components. If the complex mixture contains compounds all boiling in a fairly narrow temperature range, no particular difficulty is presented. If the most and least volatile compounds in the mixture differ greatly in boiling point (the more common case), a key compound may be chosen somewhere in the middle range. The finally obtained true boiling-point curve, however, may be expected to have so low an accuracy on its ends as to require a repetition of the calculation with other compounds chosen as key compounds, on the low-boiling ends of the curve, respectively.

With the key compound chosen its vapor pressure curve, P vs. T, must be constructed from available data. Since the ratio of the vapor pressures at the limiting temperature of interest may be as great as 1000, it may be desirable to plot $\log P$ vs. T. If vapor pressure data on the particular key compound chosen are limited, one of several methods may be used for establishing the form of the $P - T$ function outside the range of data. A plot, for example of $\log P$ vs. $1/T$, should be approximately a straight line.

With the $Y - P$ relation established, the pair of equations may be used to obtain $f(x)$, or α vs. x. There then comes the problem of interpreting α, and this is related to the assumption, not quite correct,

that the value of α for any one compound is a constant, once the key compound is fixed upon. Since α is not exactly constant, it does not exactly identify a compound. A numerical example may make the discussion clearer. Consider a mixture of paraffin hydrocarbons and let the key compound be decane. The value of α for pentane will be 590, 244, 111, 65 at temperatures 5, 35, 65, 95 degrees. This, offhand, would seem to deny the justifiability of the assumption that α is a constant for a compound. However, although some butane will appear in the first drop distilled over at the initial temperature and some in the last drop at the final temperature, most of it will distill over early in the distillation over a fairly narrow temperature range, and the value of α for butane will not vary much during the distillation of the major portion of it. A fair assumption is that its α should be evaluated at its boiling point, 33°.

Since α, for mixtures obeying Raoult's law, is the vapor pressure of the compound under consideration divided by that of the key compound chosen, and since we have decided to evaluate at the boiling temperature of a compound it follows that

$$\alpha = \bar{P}/P$$

in which P is the vapor pressure of the key compound at the boiling temperature of the compound under consideration. The value of P has already been plotted as a function of temperature—this plot is consequently available for converting the $x - P$ relation to the desired $x - T$ relation, or true boiling-point curve.

Other difficulties will arise when the compounds present in a mixture are not all members of the same hydro carbon family and consequently do not have vapor pressures progressing in a simple manner with molecular weight—when the vapors deviate from the perfect gas laws—when constant boiling mixtures are present, etc. Further consideration of these points at the present time, however, is perhaps premature.

At this point, a page of information from tables is introduced.

By means of Table I, choosing nonane for key, we can draw a curve P vs. T, Graph I*. We also have given a Simple Distillation Curve T vs. Y, Graph II*. Using Graph I, with Graph II we may get a Y vs. P curve Graph III.[1]

If we let $m = 1 - Y$, we may draw Graph IV* which is

$$m \text{ vs. } \frac{P(m) - P(0)}{m}$$

[1] See graphs at the end of paper.

TABLE I

Vapor Pressure Data on Some Pure Hydrocarbons, Anyone of Which May Be Chosen as Key Component

T_1°c	P, mm mercury		
	C_5H_{12}	C_9H_{20}	$C_{14}H_{28}$
5	220	1.3	
10	277	1.87	
20	410	3.6	
30	590	6.4	
40	860	11.2	
60	1570	25.5	
80	2800	73.	1.4
100	4500	155.	4.
120	6700	300.	10.
140	9900	555.	24.
160	13900	930.	50.
180	19000	1500.	98.

Now using equation (3.1) page 6 and differentiating with respect to the parameter F

$$\frac{dm}{dF} = \int_0^1 f(x)\, F^{f(x)-1}\, dx$$

which together with equation (4.1) page 6 yields

$$\frac{P}{\overline{P}} = \frac{m}{F\dfrac{dm}{dF}}$$

From (3.1)

(3.12) $$m = \int_0^1 F^{f(x)}\, dx \qquad (m = 1 - Y \text{ pg. 13})$$

and $m = 1$ when $F = 1$.
Hence using (4.12)

(4.13) $$1/\overline{P} \int_1^m P(m)\, dm/m = \int_1^F dF/F$$

Subtracting equal quantities from both sides we form the equation

$$1/\bar{P} \int_1^{m_1} [P(m)/m - P(0)/m]\, dm = \int_1^{F} dF/F - P(0)/\bar{P} \int_1^{m_1} dm/m. \tag{4.14}$$

If we settle upon a particular value of m, say m_1, on graph IV, the left hand member of (4.14) may be obtained by Simpson's rule by getting the area on graph IV.

The right hand member of (4.14), after integrating becomes

$$\ln F - P(0)/\bar{P} \ln m_1$$

so now we can draw m vs. F, Graph V.

Due to the fact that we now actually have the m vs. F curve we may write (3.12) more specifically as

$$m(F) = \int_0^1 F^{f(x)}\, dx \tag{3.14}$$

Let $u = f(x)$, then $du/dx = f'(x)$.
Let $U(u) = 1/f'(x)$ then $dx = U(u)\, du$ and from (3.14) we have

$$m(F) = \int_{f(0)}^{f(1)} F^u U(u)\, du \tag{3.15}$$

Let $\ln F = -v$, and let $M(\pm v) = m(e^{-v})$ and we have

$$M(v) = \int_{f(0)}^{f(1)} e^{-uv} U(u)\, du \tag{3.16}$$

Working on (3.16) and putting the following substitutions into effect,

$$u = e^{-\zeta}$$
$$U(u) = U(e^{-\zeta}) \quad -\ln f(0) < \zeta < -\ln f(1)$$
$$U(e^{-\zeta}) = 0 \qquad \zeta < -\ln f(0)$$
$$\zeta > -\ln f(1)$$
$$du = -e^{-\zeta}\, d\zeta$$
$$u = f(1), \qquad \zeta = -\ln f(1)$$
$$u = f(0), \qquad \zeta = -\ln f(0)$$
$$v = e^{\xi}, \qquad \text{we have}$$

$$e^{\xi} M(e^{\xi}) = \int_{-\infty}^{\infty} e^{-e^{\xi-\zeta}} U(e^{-\zeta})[-e^{\xi-\zeta}]\, d\zeta \tag{3.17}$$

which we compare with

$$K(\xi) = \int_{-\infty}^{\infty} K_1(\xi - \zeta)\, R(\zeta)\, d\zeta \tag{4}$$

Hence $R(\zeta) = [-U(e^{-\zeta})]$

Now expanding in Hermitian polynomials,[2] let

$$K(\xi) = e^{\xi} M(e^{\xi}) = \sum_0^9 a_n \psi_n(x) \tag{5}$$

where

$$a_n = \int_{-\infty}^{\infty} f(\xi)\psi_n(\xi)\, d\xi = \int_{-\infty}^{\infty} f(\xi)\, \frac{\phi_n(\xi)}{[2^n n!\sqrt{\pi}]^{\frac{1}{2}}}\, d\xi \tag{6}$$

$$\psi_n(\xi) = \frac{e^{-\frac{\xi^2}{2}} H_n(\xi)}{[2^n n!\sqrt{\pi}]^{\frac{1}{2}}} \tag{7}$$

$$\phi_n(\xi) = e^{-\frac{\xi^2}{2}} H_n(\xi) \tag{8}$$

$$H_n(\xi) = 2^n\left[\xi^n - \frac{n(n-1)}{4}\,\xi^{n-2} + \frac{n(n-1)(n-2)(n-3)}{4(8)}\,\xi^{n-4} - \cdots\right] \tag{9}$$

The $f(\xi)$ of equation (6) is the function to be approximated. In our case $f(\xi)$ is $e^{\xi} M(e^{\xi})$

From (5) and (3.17)

$$\sum_0^9 a_n \psi_n(\xi) = \int_{-\infty}^{\infty} e^{-e^{\xi-\zeta}} e^{\xi-\zeta}[-U(e^{-\zeta})]\, d\zeta \tag{3.18}$$

Taking the Fourier transform of both sides

$$\sum_0^9 \frac{a_n}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iu\xi}\psi_n(\xi)\, d\xi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iu\xi}\, d\xi \int_{-\infty}^{\infty} e^{-e^{\xi-\zeta}} e^{\xi-\zeta}[-U(e^{\zeta})]\, d\zeta$$

but

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iu\xi}\psi_n(\xi)\, d\xi = i^n \psi_n(u)$$

$$\sum_0^9 a_n i^n \psi_n(u) = 1/\sqrt{2\pi} \int_{-\infty}^{\infty} [-U(e^{-\zeta})]\, d\zeta \int_{-\infty}^{\infty} e^{iu\xi} e^{-e^{\xi-\zeta}} e^{\xi-\zeta}\, d\xi$$

[2] For the properties of Hermitian polynomials and Fourier transforms used here and later see for instance: Norbert Wiener, The Fourier Integral, Cambridge, 1933, Chapter I.

Let $\xi = \bar{\xi} + \zeta$

$$\sum_0^\infty a_n i^n \psi_n(u) = 1/\sqrt{2\pi} \int_{-\infty}^{\infty} [-U(e^{-\bar{\xi}})] e^{iu\zeta}\, d\zeta \int_{-\infty}^{\infty} e^{iu\bar{\xi}} e^{-e^{\bar{\xi}}} e^{\bar{\xi}}\, d\bar{\xi}$$

but $$\int_{-\infty}^{\infty} e^{iu\bar{\xi}} e^{-e^{\bar{\xi}}} e^{\bar{\xi}}\, d\bar{\xi} = \int_0^\infty dz\, z^{iu} e^{-z} = \Gamma(iu+1)$$

where $z = e^{\bar{\xi}}$.

Let $$\phi(u) = 1/\sqrt{2\pi} \int_{-\infty}^{\infty} [-U(e^{-\bar{\xi}})] e^{iu\zeta}\, d\zeta = 1/\sqrt{2\pi} \int_{-\infty}^{\infty} R(\zeta) e^{iu\zeta}\, d\zeta$$

$$\sum_0^\infty a_n i^n \psi_n(u) = \Gamma(iu+1) \cdot \phi(u)$$

or $$\phi(u) = \frac{\sum_0^\infty a_n i^n \psi_n(u)}{\Gamma(iu+1)}$$

Proceeding

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iu\zeta} \phi(u)\, du = \sum_0^\infty a_n i^n \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\psi_n(u) e^{-iu\zeta}}{\Gamma(iu+1)}\, du$$

And

$$\sum_0^\infty a_n i^n \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\psi_n(u)}{\Gamma(iu+1)} e^{-iu\zeta}\, du = R(\zeta)$$

Let $$F_n(\zeta) = \frac{i^n}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\psi_n(u)}{\Gamma(iu+1)} e^{-iu\zeta}\, du$$

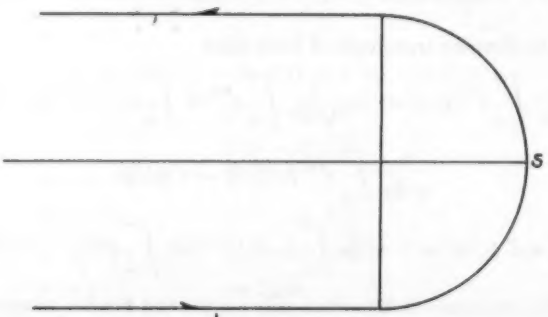

FIG. 2

but

$$\frac{1}{\Gamma(iu+1)} = \frac{1}{2\pi i}\int_L e^s S^{-iu-1}\,ds$$

$$F_n(\zeta) = \frac{i^n}{\sqrt{2\pi}}\int_{-\infty}^{\infty} du\,\psi_n(u)e^{-iu\zeta}\,\frac{1}{2\pi i}\int_L e^s S^{-iu-1}\,ds$$

$$= \frac{i^n}{2\pi i}\int_L ds\,e^s s^{-1}\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} du\,\psi_n(u)s^{-iu}e^{-iu\zeta}$$

$$W = \log s$$

$$= \log|S| + i\,\text{ang}\,s \qquad s = e^W$$

$$F_n(\zeta) = \frac{i^n}{2\pi i}\int_{L_1} dw\,e^{eW}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} du\,\psi_n(u)e^{-iu(\zeta+W)}$$

$$= \frac{i^n}{2\pi i}\int_{L_1} dw\,e^{eW}\,(-i)^n\psi_n(\zeta+W)$$

$$F_n(\zeta) = \frac{1}{2\pi i}\int_{L_1} dW\,e^{eW}\,\psi_n(\zeta+W)$$

$$F_n(\zeta) = \frac{1}{2\pi i}\int_{\infty}^{-\infty} du\,e^{-e^u}\psi_n(\zeta+u-\overset{\circ}{\pi i})$$

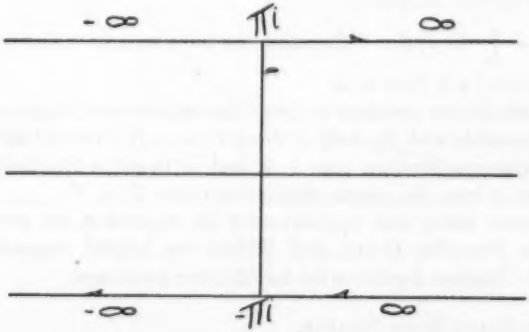

FIG. 3

$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty} du\, e^{-e^u}\, \psi_n(\zeta + u + \pi i)$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} du\, e^{-e^u}\, \frac{1}{2i}\,[\psi_n(\zeta + u + \pi i) - \psi_n(\zeta + u - \pi i)]$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} du\, e^{-e^u}\, \mathbf{I}[\psi_n(\zeta + u + \pi i)]$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} du\, e^{-e^{\bar{u}} - \zeta}\, \mathbf{I}[\psi_n(\bar{u} + \pi i)]$$

$$R(\zeta) = \sum_{0}^{9} a_n F_n(\zeta) \qquad F_n(\zeta) \quad -8 \leqq \zeta \leqq 8$$

Our problem is practically finished now for a table may be resorted to giving values of $F_n(x)$ for various values of x where

$$F_n(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} du\, e^{-e^{u-x}}\, \mathbf{I}[\psi_n(u + \pi i)]$$

Hence one can compute the $a_0, a_1 \cdots a_n$ by the Hermitian formulae as in the case of $e^{\frac{1}{2}}K(\xi)$, then take values of $F_n(x)$ from the table and substitute in

$$R(\zeta) = \sum_{0}^{9} a_n F_n(\zeta)$$

thus getting the $R(\zeta)$ vs. (ζ) curve.
But $R(\zeta) = -U(u) = -U(e^{-\zeta})$.
So using the fact that $R = -U(u)$, we have a $U(u)$ vs. ζ curve.

Next using the fact that $u = e^{-\zeta}$, we have a $U(u)$ vs. $e^{-\zeta}$ or in other words a $U(u)$ vs. u curve.

But $x = \int_0^u U(u)\, du$. Therefore, we have an x vs. u Curve, which is x vs. $f(x)$ since $u = f(x) = \alpha$.

This concludes our problem as far as the mathematics is concerned and it is now possible with the help of this x vs. $\alpha = f(x)$ curve together with the procedure outlined on page 7, 9, and 10 to get a true boiling point curve T vs. x from the simple distillation curve Y vs. T.

The writer takes this oppportunity of expressing his gratitude to Professors Franklin, Hottel and Wiener for helpful suggestions, and also to Dr. Norman Levinson for his effective assistance.

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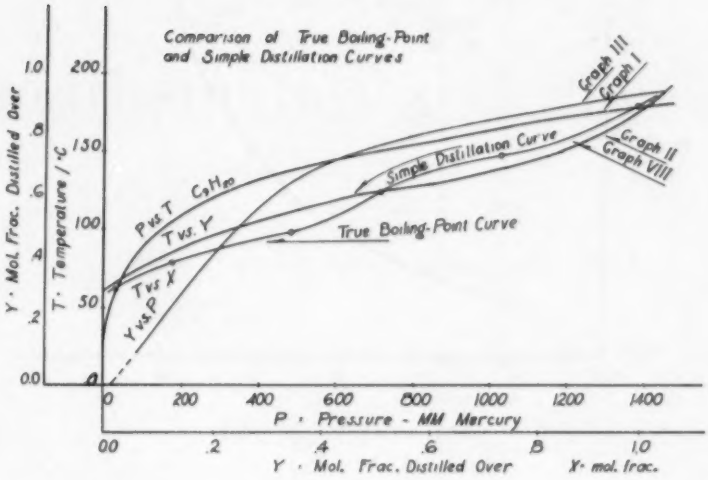
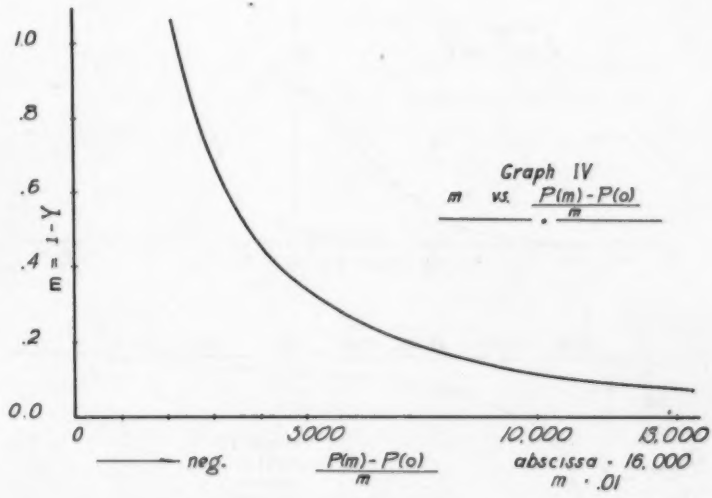
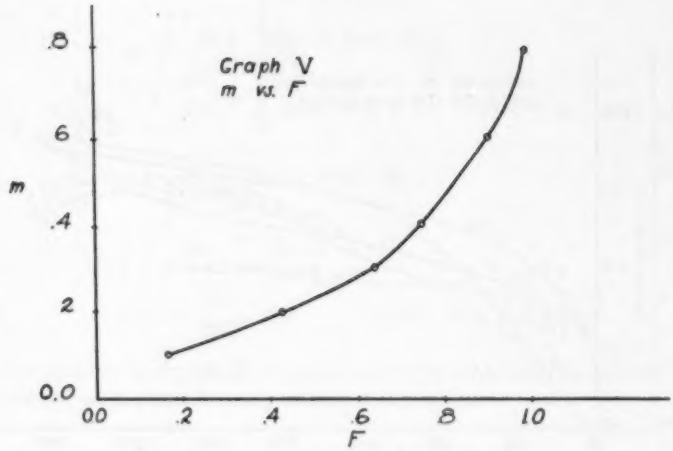
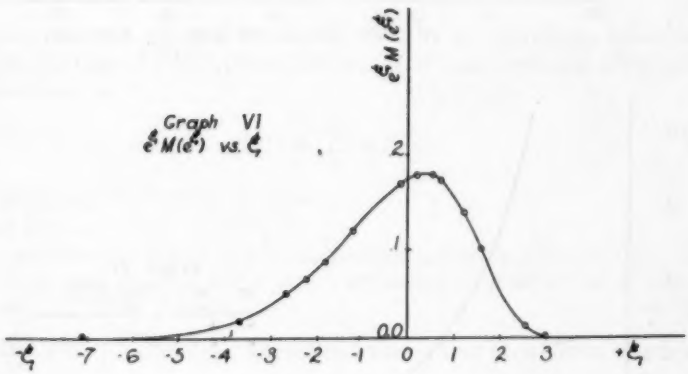
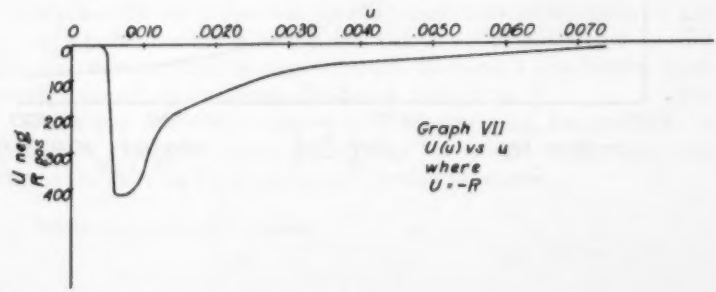

Comparison of True Boiling-Point and Simple Distillation Curves
Graph III
Graph I
Graph II
Graph VIII
Simple Distillation Curve
True Boiling-Point Curve
P vs T C6H6
T vs Y
T vs X
Y vs P
Y · Mol. Frac. Distilled Over
T · Temperature / °C
P · Pressure - MM Mercury
Y · Mol. Frac. Distilled Over
Y· mol. frac.


Graph IV
m vs. P(m) - P(o) / m
m = 1 - Y
neg.
P(m) - P(o) / m
abscissa · 16.000
m · .01


Graph V
m vs. F
m
F
.8
6
.4
2
0.0
00
.2
.4
.6
.8
1.0


Graph VI
$\xi^4 M(\xi^2)$ vs. ξ
$\xi^4 M(\xi^2)$
2
1
0.0
$-\xi$
-7
-6
-5
-4
-3
-2
-1
0
1
2
3
$+\xi$


u
0
.0010
.0020
.0030
.0040
.0050
.0060
.0070
U neg
R
100
200
300
400
Graph VII
U(u) vs u
where
U = -R